

Is Electromagnetic Field Momentum Due to the Flow of Field Energy?*

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Conservation laws such as the Poynting theorem of electrodynamics that are based on the divergence of a second-rank four-tensor, are fundamentally different from conservation laws such as the conservation of electric charge that are based on the divergence of a four-vector. This article investigates the consequences of this difference for understanding the relation between electromagnetic field momentum and the flow of electromagnetic field energy.

Momentum and energy conservation require electromagnetic field momentum and energy to be treated as physically real, even in static fields. This motivates the conjecture that field momentum might be due to the flow of a relativistic mass density (defined as energy density divided by the square of the speed of light).

This article investigates the velocity of such an energy flow and finds a conflict between two different definitions of it, both of which originally seem plausible if the flow is to be taken as real. This investigation is careful to respect the transformation rules of special relativity throughout.

The paper demonstrates that the consensus definition of the flow velocity of electromagnetic energy is inconsistent with the transformation rules of special relativity, and hence is incorrect. A correct flow velocity is then derived which is completely consistent with those transformation rules.

The conclusion is that these conflicting definitions of energy flow velocity cannot be resolved in a way that is consistent with special relativity and that also allows electromagnetic field momentum density to be the result of relativistic mass flow. Though real, field momentum density cannot be explained as the flow of field energy.

As a byproduct of the study, it is also shown that there is a comoving system in which the electromagnetic energy-momentum tensor is reduced to a simple diagonal form, with two of its diagonal elements equal to the energy density and the other two diagonal elements equal to plus and minus a single parameter derived from the electromagnetic field values, a result that places constraints on possible fluid models of electromagnetism.

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1 Introduction

The Poynting theorem of electrodynamics is based on the divergence of the second-rank energy-momentum tensor $T^{\alpha\beta}$. Conservation laws based on the divergence of a second-rank tensor are fundamentally different from conservation laws, such as the conservation of electric charge, that are based on the divergence of a four-vector. This article investigates the consequences of this difference for understanding the relation between electromagnetic field momentum and the flow of electromagnetic field energy.

The example of a rotating disk with a magnet at its center and charged spheres on its perimeter provides a convincing argument that, to preserve the principle of angular momentum conservation, the field momentum of even a static electromagnetic field must be considered physically real.¹ It is also generally assumed that conservation of energy requires the energy density of the electromagnetic field to be physically real, even for static fields. This article accepts the reality of the field momentum and energy, but questions the flow of field energy as the source of field momentum.

Explanation of electromagnetic field momentum as an energy flow depends crucially on a *correct* definition for the velocity of that energy flow. Since special relativity is the invariance theory of electromagnetism, throughout this paper by *correct* or *valid* we will mean that a definition or construct is correct or valid only if it is consistent with special relativity.

We give two possible definitions for the velocity of electromagnetic energy flow, which we refer to as Definitions A and B. For conservation laws such as charge conservation that are derived from the divergence of a four-vector, these two definitions coincide. But for conservation laws such as the Poynting theorem based on the divergence of the second-rank energy-momentum tensor, the two definitions differ and only one can be correct.

Definition A: In the modern, post-relativity era it has been the consensus in the literature, at least since the first English edition of Born and Wolf's *Optics*,² that the electromagnetic energy flow velocity is $\mathbf{V}_A = \mathbf{S}/\mathcal{E}$ where \mathbf{S} is the Poynting vector and \mathcal{E} is the energy density. We refer to this as Definition A.

With Definition A, clearing the fraction to $\mathbf{S} = \mathcal{E}\mathbf{V}_A$ and dividing by c^2 gives $\mathbf{G} = \mathcal{M}_{\text{rel}}\mathbf{V}_A$ where $\mathbf{G} = \mathbf{S}/c^2$ is the mo-

¹Feynman et al [6], Section 17-4, Section 27-6, and Figure 17-5. Quantitative matches of field to mechanical angular momentum are found, for example, in Romer [16] and Boos [2].

²In a discussion of the Poynting theorem in material media, but with no special attention to Lorentz covariance, Born and Wolf [3] Section 14.2, eq.(8) identify \mathbf{V}_A as the *velocity of energy transport or ray velocity*. (The first edition of Born and Wolf's text appeared in 1959.) Section B.2 of Smith [19] echoes Born and Wolf but provides no new derivation. Geppert [7] makes the same identification. More recently, Sebens [17, 18] relies on these and other sources to identify \mathbf{V}_A as the electromagnetic mass flow velocity. Sebens also considers earlier, pre-Einstein studies by Poincare. The present paper however is focussed on the reconsideration of the subject forced by special relativity.

mentum density, and $\mathcal{M}_{\text{rel}} = \mathcal{E}/c^2$ is the so-called relativistic mass density. Thus Definition A implies that the momentum density is due to flow of relativistic mass.

This article investigates this consensus claim and finds reason to doubt it. Sections 2 and 3 demonstrate that the coordinate flow velocity \mathbf{V}_A is not *correct* in the above sense. It is not consistent with the transformation rules of special relativity.

Definition B: The velocity of the energy (mass) flow at a given event can also be defined as the velocity of an observer who measures the Poynting energy flux vector to be zero at that event. If \mathbf{S} truly is the flux of energy flow, then an observer comoving with this flow should observe a zero value of that energy flux. This is Definition B and its velocity will be denoted as \mathbf{V}_B . An explicit and relativistically correct derivation of \mathbf{V}_B is presented in Section 4. Its possible caveats are discussed in Section 5 and Section 6.

Section 7 demonstrates that the comoving reference system used in the derivation of Definition B allows a reduction of the electromagnetic energy-momentum tensor to a simple, diagonal form, with two of its diagonal elements equal to the energy density in the comoving frame and the other two diagonal elements equal to plus and minus a single parameter derived from the electromagnetic field values. This reduction of the electromagnetic energy-momentum tensor is shown to place important constraints on fluid-dynamic models of energy flow in the electromagnetic field.

Section 8 concludes that electromagnetic field momentum density is not due to the flow of an electromagnetic mass density. Since $\mathbf{V}_A \neq \mathbf{V}_B$, a choice between Definitions A and B must be made. As noted above, velocity \mathbf{V}_A would show electromagnetic field momentum to result from the flow of field energy. However \mathbf{V}_A is not relativistically correct and must be rejected. However, with the correct and relativistically valid choice \mathbf{V}_B , momentum density *cannot* be explained as due to the motion of a relativistic mass density $\mathcal{M}_{\text{rel}} = \mathcal{E}/c^2$. Thus the title question of this paper has a negative answer.

Section 9 accepts the negative results of the present study of relativistic mass flow, and speculates about possible ways forward in the search for a model, if any, of physics that might underly the Maxwell Equations.

This paper uses Heaviside-Lorentz units. We denote four-vectors as $\mathbf{K} = K^0\mathbf{e}_0 + \mathbf{K}$ where \mathbf{e}_0 is the time unit vector and the three-vector part is understood to be $\mathbf{K} = K^1\mathbf{e}_1 + K^2\mathbf{e}_2 + K^3\mathbf{e}_3$. In the Einstein summation convention, Greek indices range from 0 to 3, Roman indices from 1 to 3. The Minkowski metric tensor is $(\eta_{\alpha\beta}) = (\eta^{\alpha\beta}) = \text{diag}(-1, +1, +1, +1)$. Three-vectors are written with bold type \mathbf{K} , and their magnitudes as K . Thus $|\mathbf{K}| = K$.

2 Arguments for Definition A

Definition A for the energy flow velocity is defined by equivalent³ formulas

$$\mathbf{S} = \mathcal{E}\mathbf{V}_A \quad \text{and} \quad \mathbf{V}_A = \mathbf{S}/\mathcal{E} \quad (1)$$

Writing $\mathbf{S} = c(\mathbf{E} \times \mathbf{B})$ and $\mathcal{E} = (E^2 + B^2)/2$, in terms of the electric and magnetic fields \mathbf{E} and \mathbf{B} , gives⁴

$$\mathbf{V}_A = \frac{\mathbf{S}}{\mathcal{E}} = \frac{2c(\mathbf{E} \times \mathbf{B})}{(E^2 + B^2)} \quad (2)$$

It is easily shown from the inequalities $(E - B)^2 \geq 0$ and $EB \geq |\mathbf{E} \times \mathbf{B}|$ that $|\mathbf{V}_A| \leq c$.

Argument from Analogy: The consensus definition that energy flow velocity is simply $\mathbf{V}_A = \mathbf{S}/\mathcal{E}$ is suggested by analogy with the well understood example of $\mathbf{V}_q = \mathbf{J}/\rho$ as the velocity of charge flow, given electric charge density ρ and charge flux density \mathbf{J} . It will therefore be useful to begin with a review of the properties of charge flow.

The charge flux four-vector is $\mathbf{J} = c\rho\mathbf{e}_0 + \mathbf{J}$. This four-vector can be timelike, spacelike, or null; but there are useful cases in which it is timelike. In these cases, the velocity $\mathbf{V}_{qA} = \mathbf{J}/\rho$ has magnitude less than the speed of light.

As noted in Section 1, there are actually two possible definitions of the velocity of charge flow. One is the $\mathbf{V}_{qA} = \mathbf{J}/\rho$ just defined. The other is the velocity \mathbf{V}_{qB} of a comoving reference system (which we will refer to as the primed system) in which the charge flux vector vanishes, $\mathbf{J}' = 0$. Since an observer at rest in the prime system sees a zero charge flux, this observer must be moving with the charge flow, and his velocity \mathbf{V}_{qB} must be the velocity of that flow.

Since $\mathbf{J} = c\rho\mathbf{e}_0 + \mathbf{J}$ is known to be a four-vector, Appendix II shows that a Lorentz boost transformation⁵ with the boost velocity $\mathbf{V} = \mathbf{V}_{qA} = \mathbf{J}/\rho$ actually *also* transforms from the original unprimed reference system to a reference system in which $\mathbf{J}' = 0$. Thus, for the case of charge flow, the two possible flow velocity definitions coincide, $\mathbf{V}_{qB} = \mathbf{V}_{qA}$.

But the argument in Appendix II depends essentially on the four-vector transformation rules of \mathbf{J} . It is crucial that $\mathbf{J} = c\rho\mathbf{e}_0 + \mathbf{J}$ is a legitimate four-vector whose components transform according to the standard rule $J'^\alpha = \Lambda^\alpha_\beta J^\beta$.

If we try to apply the reasoning of Appendix II with ρ, \mathbf{J} replaced by \mathcal{E}, \mathbf{S} , the chain of logic does not go through to its conclusion. Unlike ρ and \mathbf{J} , the energy density \mathcal{E} and energy-flux vector \mathbf{S} of the Poynting theorem do *not* transform as components of a four-vector. The transformation rule for $c\mathcal{E}$ and $(\mathbf{S})_i$ is as the (00) and (0i) components of the more complicated expression given in eq.(6) below, which will also involve contributions from cM_{ij} terms. Instead of $J'^\alpha = \Lambda^\alpha_\beta J^\beta$

³Since \mathcal{E} is nonzero except when $E=B=0$, throughout this paper we will take Definition A, $\mathbf{V}_A = \mathbf{S}/\mathcal{E}$, to be equivalent to the formula $\mathbf{S} = \mathcal{E}\mathbf{V}_A$.

⁴Electromagnetic formulas in this paper are taken from Griffiths [8] and Jackson [9], with translation into Heaviside-Lorentz units.

⁵The Lorentz boost formalism is summarized in Appendices I and I.1.

we have $S'^\alpha \neq \Lambda^\alpha_\beta S^\beta$. The argument in Appendix II thus fails when ρ, \mathbf{J} are replaced by \mathcal{E}, \mathbf{S} . The boost velocity \mathbf{V}_B that would make $\mathbf{S}' = 0$ does not coincide with \mathbf{V}_A .

The analogy between charge flow and energy flow is therefore broken, and cannot be used as an argument for the consensus velocity definition $\mathbf{V}_A = \mathbf{S}/\mathcal{E}$.

In the case of the Poynting theorem, $\mathbf{V}_B \neq \mathbf{V}_A$ and both cannot be correct. A choice must be made between them. It will be shown in Section 3 that energy flow velocity \mathbf{V}_A is inconsistent with the transformation rules of special relativity and must be rejected. A correct and relativistically legitimate velocity \mathbf{V}_B is derived in Section 4.

Argument from Geometry: In addition to the above analogy with charge density, the following simple geometric construction can be used to argue for the consensus definition $\mathbf{V}_A = \mathbf{S}/\mathcal{E}$.

Geometry of a Flow: Given a flowing substance with density κ and flux density vector \mathbf{K} , define a velocity as $\mathbf{v} = \mathbf{K}/\kappa$.

Now we must examine this velocity definition \mathbf{v} to see whether it passes the test of compatibility with the rules of special relativity. If it does not, then application of the results of this inset will be a misapplication, and would lead to results inconsistent with the special theory of relativity. The test must be applied on a case-by-case basis. Some applications will be seen to be correct, but others will be misapplications.

Assuming that \mathbf{v} passes that test, the following simple geometric argument may be made. Consider an arbitrarily oriented area element $d\mathbf{a}$ and a time increment dt . The product $d\tau = (\mathbf{v} dt) \cdot d\mathbf{a}$ is a volume element. All points in $d\tau$ moving with velocity \mathbf{v} will flow through $d\mathbf{a}$ in time dt . Now multiply by κ to obtain $\kappa d\tau = \kappa \mathbf{v} \cdot d\mathbf{a} dt$, the amount of substance in $d\tau$. If we assume that all of the substance in $d\tau$ is moving with the same velocity \mathbf{v} , then $\mathbf{v} \cdot d\mathbf{a} dt$ is the amount of substance flowing through $d\mathbf{a}$ in time dt . But this amount is also, by definition of the flux density \mathbf{K} , given by $\mathbf{K} \cdot d\mathbf{a} dt$. Thus

$$\mathbf{K} \cdot d\mathbf{a} dt = \kappa \mathbf{v} \cdot d\mathbf{a} dt \quad (3)$$

Since $d\mathbf{a}$ and dt are arbitrary, it follows that $\mathbf{K} = \kappa \mathbf{v}$.

But the above assumption that all elements of the substance are moving with the same velocity \mathbf{v} is often unjustified. (Think of a flow of electrical charges with some thermal velocity.) Then the above simple geometric argument fails.

But if the argument in eq.(3) fails (while still assuming that \mathbf{v} passes the relativity test above) we can consider $\mathbf{v} = \mathbf{K}/\kappa$ to be a definition of an average flow velocity. Then $\mathbf{K} = \kappa \mathbf{v}$ is true by definition.

If this geometrical argument with $\kappa = \mathcal{E}, \mathbf{K} = \mathbf{S}$ passed the test of compatibility with relativity, it would predict that the energy flow velocity must be $\mathbf{v} = \mathbf{S}/\mathcal{E} = \mathbf{V}_A$, the consensus definition of energy flow velocity defined in eq.(2), either as a geometric construction or as a definition. However, it will be shown in the next section that $\mathbf{v} = \mathbf{S}/\mathcal{E} = \mathbf{V}_A$ does not pass the test of compatibility with relativity.

3 Caveats of Definition A

The principal difficulty with Definition A for the energy flow velocity is that it is inconsistent with the transformation rules of special relativity.

We take a flow velocity definition to be *relativistically valid* only if that definition passes a simple test using the Einstein velocity addition formula. Since that test is derived directly from the transformation rules of special relativity, any flow velocity definition that fails the test must also violate some rule of special relativity. Definition A fails this simple test and hence is not relativistically valid.

Einstein Addition Test: Consider two different but parallel initial reference systems referred to as the unprimed and asterisk systems. Let the asterisk system be obtained from the unprimed one by a boost transformation with boost velocity $\mathbf{V} = V\mathbf{e}_1$ of arbitrary magnitude V . Then unit vectors \mathbf{e}_i are parallel to the corresponding \mathbf{e}_i^* , for $i = 1, 2, 3$. Let the velocity of a comoving observer moving with the energy flow be \mathbf{v} relative to the unprimed system, and the velocity of that same observer relative to the asterisk system be \mathbf{v}^* . With no loss of generality, the unprimed system can be oriented so that \mathbf{v} is in the \mathbf{e}_1 direction. Then $\mathbf{v} = v\mathbf{e}_1$ and hence $\mathbf{v}^* = v^*\mathbf{e}_1^*$.

Electrodynamics in vacuum except for a possible explicit source can be expressed in manifestly covariant form and therefore must be true regardless of the choice of initial reference system; there can be no privileged initial reference system. So any relativistically correct derivation that defines flow velocity \mathbf{v} when applied in the unprimed system can also be applied to define flow velocity definition \mathbf{v}^* when applied in the asterisk system. And these two velocities are velocities of the same comoving observer. Thus, when we make a boost transformation with boost velocity $\mathbf{V} = V\mathbf{e}_1$ between the unprimed and asterisk systems, these velocities \mathbf{v} and \mathbf{v}^* must transform by the Einstein velocity-addition formula (See Example 12.6 of Griffiths[8])

$$(v/c) = \frac{(V/c) + (v^*/c)}{1 + (Vv^*/c^2)} \quad (4)$$

This is the Einstein Addition Test that any relativistically valid flow velocity definition must pass.

Note that this test is a necessary condition for consistency with the transformation rules of special relativity. Any electromagnetic energy flow definition that fails to pass the Einstein Addition Test in eq.(4) must necessarily violate the transformation rules of special relativity from which eq.(4) is directly derived, and therefore must be rejected.

We now show that the consensus energy-flow velocity definition $\mathbf{V}_A = \mathbf{S}/\mathcal{E}$ fails the Einstein Addition Test. This failure is due to the fact that $c\mathcal{E}$ and \mathbf{S} are not components of a four-vector. Unlike the legitimate four-vector of charge flow $\mathbf{J} = c\rho\mathbf{e}_0 + \mathbf{J}$, there is no four-vector $\mathbf{S} \propto c\mathcal{E}\mathbf{e}_0 + \mathbf{S}$. (I use the symbol \propto to remind the reader that, though written formally as a four-vector here, it does not actually transform

as one.)⁶ Noting that $\mathbf{S} = c^2\mathbf{G}$ where \mathbf{G} is the linear momentum density of the electromagnetic field, the $c\mathcal{E}$ and $(\mathbf{S})_i$ actually transform as the (00) and (0i) components of a four-tensor $cT^{\alpha\beta}$ defined as c times the standard electromagnetic energy-momentum tensor

$$(T^{\alpha\beta}) = \begin{pmatrix} \mathcal{E} & cG_1 & cG_2 & cG_3 \\ cG_1 & M_{11} & M_{12} & M_{13} \\ cG_2 & M_{21} & M_{22} & M_{23} \\ cG_3 & M_{31} & M_{32} & M_{33} \end{pmatrix} \quad (5)$$

where $M_{ij} = -(E_iE_j + B_iB_j) + \frac{1}{2}\delta_{ij}(E^2 + B^2)$. Thus the transformation rule for $c\mathcal{E}$ and $(\mathbf{S})_i$ is as the (00) and (0i) components of the more complicated expression

$$cT'^{\alpha\beta} = \Lambda^\alpha_\mu \Lambda^\beta_\nu cT^{\mu\nu} \quad (6)$$

which will also involve contributions from the cM_{ij} terms.

To see that energy-flow velocity definition $\mathbf{V}_A = \mathbf{S}/\mathcal{E}$ fails the Einstein Addition Test, begin with the example of the charge flow definition $\mathbf{V}_{qA} = \mathbf{J}/\rho$ that passes the test.

Appendix III demonstrates that the coordinate velocities $\mathbf{V}_{qA} = \mathbf{J}/\rho$ and $\mathbf{V}_{qA}^* = \mathbf{J}^*/\rho^*$ derived from the charge density four-vector \mathbf{J} do pass the Einstein Addition Test, as must be true for any well-defined coordinate velocity. When the inverse four-vector transformation rule $J^\alpha = \underline{\Lambda}^\alpha_\beta J^{*\beta}$ is used to write J^0 and J^i in terms of \mathbf{V} and the asterisk system quantities J^{*0} and J^{*i} , the result is the last expression in eq.(38), which agrees with the Einstein velocity addition rule eq.(4) and hence with special relativity.

However, if we now attempt to apply this same argument to the case of $\mathbf{V}_A = \mathbf{S}/\mathcal{E}$ and $\mathbf{V}_A^* = \mathbf{S}^*/\mathcal{E}^*$, the argument of Appendix III fails. In this case, the equality $J^\alpha = \underline{\Lambda}^\alpha_\beta J^{*\beta}$ is replaced by the inequality $S^\alpha \neq \underline{\Lambda}^\alpha_\beta S^{*\beta}$ resulting from the failure of \mathbf{S} to be a four-vector. Thus, eqs.(37, 38) are not true when \mathbf{J} is replaced by \mathbf{S} , and the argument does not go through to its conclusion.

In place of the equality in eq.(38) for the charge density case, in the case of \mathbf{V}_A we have the inequality

$$(V_A/c) = (S/\mathcal{E}c) \neq \frac{(V/c) + (V_A^*/c)}{1 + (V V_A^*/c^2)} \quad (7)$$

where the expression on the extreme right in eq.(7) would be the correct Einstein velocity addition result. Thus \mathbf{V}_A fails the Einstein Addition Test.

⁶A related point is made by Rohrlich [15], using the so-called von Laue's theorem to argue that integrals of $c\mathcal{E}$ and \mathbf{S} over hyperplanes may in some cases transform as four-vectors. But we are treating these quantities locally, at a particular event. Von Laue's theorem does not imply that the local field functions $c\mathcal{E}$ and \mathbf{S} (the integrands of these hyperplane integrals) themselves transform as components of a four-vector. They do not. See also Chapter 6 of Rohrlich [14].

The inequality in eq.(7) can also be derived directly from the transformation rules for the E and B fields, without making any reference to the charge flow analogy. Using the same geometry as in the *Einstein Addition Test* above, and a transformation rule similar to eq.(10), it can be shown that

$$(V_A/c) = \frac{(V/c) + (1 + V^2/c^2)(V_A^*/c)}{(1 + V^2/c^2) + 2(V V_A^*/c^2)} \neq \frac{(V/c) + (V_A^*/c)}{1 + (V V_A^*/c^2)} \quad (8)$$

which corroborates the inequality in eq.(7) and shows again that $\mathbf{V}_A = \mathbf{S}/\mathcal{E}$ fails the *Einstein Addition Test*.

The results of the present Section can be stated as the following proposition:

Proposition 1: *The consensus definition of energy flow velocity*

$$\mathbf{S} = \mathcal{E}\mathbf{V}_A \quad \text{and} \quad \mathbf{V}_A = \mathbf{S}/\mathcal{E} \quad (9)$$

defines a velocity \mathbf{V}_A that fails the Einstein Addition Test and therefore cannot be used as a relativistically valid definition for the electromagnetic energy flow velocity.

In spite of its relativistic incorrectness, the definition in eq.(9) might seem to be proved by the geometrical argument in *Geometry of a Flow* from Section 2. However, note that the success of that geometric argument depends essentially on the assumption that all elements of the flowing substance are moving with the same velocity. But the Poynting theorem gives us \mathcal{E} and \mathbf{S} as *densities* and not as precise values. There is no reason to suppose that \mathcal{E} and \mathbf{S} are densities of a set of elements all moving at exactly the same velocity. In fact, the relativistic incorrectness of eq.(9) argues that they are not.

In summary, regardless of how it is derived, either from a flawed analogy with charge flow, or from a misapplication of *Geometry of a Flow*, the definition $\mathbf{V}_A = \mathbf{S}/\mathcal{E}$ of energy flow velocity violates the transformation rules of special relativity and is not relativistically valid. It follows that \mathbf{V}_A cannot be the velocity of electromagnetic energy flow in a relativistically correct theory.

4 Detail of Definition B

Definition B of the energy flow velocity, denoted \mathbf{V}_B , is the velocity of a comoving observer who measures a zero energy flux. Expressed in the precise language of Lorentz boost transformations:

The coordinate velocity of the flow of electromagnetic field energy at a given event is the velocity \mathbf{V}_B of a Lorentz boost that transforms the original reference system into a reference system in which the Poynting energy flux vector is zero at that event.

An observer at that event and at rest in this transformed system, which we call the comoving system and denote by primes, therefore measures a zero energy flux. The zero flux measurement indicates that this observer is comoving with the flow of energy. Such an observer has coordinate velocity

\mathbf{V}_B relative to the original system,⁷ and therefore \mathbf{V}_B is the coordinate velocity of the energy flow at the given event.

The problem is to find this boost velocity \mathbf{V}_B . An analogous problem arises in the generic theory of relativistic fluid flow.⁸ There a velocity can be defined as $\mathbf{V}_a = \mathbf{p}c^2/e$ analogous to our $\mathbf{V}_A = \mathbf{S}/\mathcal{E} = \mathbf{G}c^2/\mathcal{E}$. But, a proof analogous to the proof in Section 3 shows that velocity to be inconsistent with the Einstein velocity relation of special relativity and hence not a valid definition. In the theory of fluid flow, there is no other way to derive a flow velocity from first principles. One solution is simply to *assert* that there *must be* a primed reference system moving with the flow even though we have been unable to derive it; to assert that the energy-momentum tensor in that system must have the isotropic form $(X'^{\alpha\beta}) = \text{diag}\{\varepsilon', \pi', \pi', \pi'\}$, where ε' is an energy density and π' is a pressure. This is called the *perfect fluid* model. However it remains true that the flow velocity and the form of the energy-momentum tensor are simply asserted rather than derived.⁹

In the electromagnetic case considered in the present paper, however, the failure of \mathbf{V}_A does not exhaust our ways of deriving \mathbf{V}_B . We can fall back on the rich structure of the Maxwell equations themselves, which underlie the definition of the energy-momentum tensor $T^{\alpha\beta}$ and from which it was derived. Thus in the electromagnetic case we are not reduced to merely *asserting* the existence of a comoving frame. We can actually *derive* the boost velocity \mathbf{V}_B and the form of the energy-momentum tensor in the comoving frame, starting from first principles.

The rules for transformation of electric and magnetic fields by a boost with velocity \mathbf{V}_B can be written in a special relativistically correct but not manifestly covariant form¹⁰

$$\begin{aligned} \mathbf{E}' &\doteq \gamma_B \left(\mathbf{E} + \frac{\mathbf{V}_B}{c} \times \mathbf{B} \right) + (1 - \gamma_B) \frac{\mathbf{V}_B (\mathbf{V}_B \cdot \mathbf{E})}{V_B^2} \\ \mathbf{B}' &\doteq \gamma_B \left(\mathbf{B} - \frac{\mathbf{V}_B}{c} \times \mathbf{E} \right) + (1 - \gamma_B) \frac{\mathbf{V}_B (\mathbf{V}_B \cdot \mathbf{B})}{V_B^2} \end{aligned} \quad (10)$$

⁷See Appendix I.2 for a demonstration that any point at rest in the primed system moves with coordinate velocity \mathbf{V}_B .

⁸Part I, Chapter 2 of Weinberg[20] presents what I will refer to as a *generic* theory. It assumes only that a fluid is composed of a countable set of small particles characterized by their mass m_n , position \mathbf{x}_n , and velocity \mathbf{v}_n . Weinberg (e.g. his eq.(2.8.1) *et seq*) uses the language of Dirac delta function densities, but his formulas are easily translated into more standard density functions.

⁹Weinberg[20] Part I, Chapter 2, Section 10, eq.(2.10.1) *et seq*. Note that Weinberg introduces the perfect fluid by saying, "A useful approximation is ..." rather than attempting to derive it from his previous work in his Chapter 2.

¹⁰See Section 11.10 of Jackson [9], eq.(11.149). The \doteq symbol means that the components of the three-vector on the left side of this symbol, expressed in the primed coordinate system, are numerically equal to the corresponding components of the three-vector on the right side of this symbol, expressed in the original unprimed system. If $\mathbf{a}' \doteq \mathbf{c}$ and $\mathbf{b}' \doteq \mathbf{d}$, it is easily proved that: (a) $(\mathbf{a}' \times \mathbf{b}') \doteq (\mathbf{c} \times \mathbf{d})$ and (b) $(\mathbf{a}' \cdot \mathbf{b}') = (\mathbf{c} \cdot \mathbf{d})$. (c) Also if $\mathbf{w}' \doteq \mathbf{w}$ then the magnitudes are equal, $w' = |\mathbf{w}'| = |\mathbf{w}| = w$.

where the Lorentz factor is $\gamma_B = (1 - V_B^2/c^2)^{-1/2}$.

The boost velocity \mathbf{V}_B can then be found by writing

$$\mathbf{V}_B = \lambda \mathbf{V}_A \quad (11)$$

where λ is a rotationally scalar quantity to be determined. The velocity \mathbf{V}_B will have the same *direction* as \mathbf{V}_A but not the same *magnitude*.

Since \mathbf{V}_A and hence \mathbf{V}_B are perpendicular to both the electric and magnetic fields, it follows that $(\mathbf{V}_B \cdot \mathbf{E}) = (\mathbf{V}_B \cdot \mathbf{B}) = 0$. Thus, eq.(10) reduces to¹¹

$$\begin{aligned} \mathbf{E}' &\doteq \gamma_B \left(\mathbf{E} + \frac{\mathbf{V}_B}{c} \times \mathbf{B} \right) \\ \mathbf{B}' &\doteq \gamma_B \left(\mathbf{B} - \frac{\mathbf{V}_B}{c} \times \mathbf{E} \right) \end{aligned} \quad (12)$$

Insert eq.(12) into the equation for the Poynting vector in the comoving system, $\mathbf{S}' = c\mathbf{E}' \times \mathbf{B}'$. Using property (a) of the symbol \doteq from footnote 10 together with eq.(11) and then eq.(2), Appendix IV demonstrates that

$$\mathbf{S}' = c\mathbf{E}' \times \mathbf{B}' \doteq \gamma_B^2 c (\mathbf{E} \times \mathbf{B}) \left((V_A/c)^2 \lambda^2 - 2\lambda + 1 \right) \quad (13)$$

Choose λ to solve the quadratic equation

$$\left((V_A/c)^2 \lambda^2 - 2\lambda + 1 \right) = 0 \quad (14)$$

Then eq.(13) makes $\mathbf{S}' = 0$, as required by Definition B. The solution is

$$\lambda = \frac{1}{(V_A/c)^2} \left\{ 1 - \sqrt{1 - (V_A/c)^2} \right\} \quad (15)$$

From eq.(11), the Definition B for the velocity of the energy flow is therefore

$$\mathbf{V}_B = \lambda \mathbf{V}_A = \frac{1}{(V_A/c)^2} \left\{ 1 - \sqrt{1 - (V_A/c)^2} \right\} \mathbf{V}_A \quad (16)$$

where \mathbf{V}_A is defined in eq.(2).

This \mathbf{V}_B is the relativistically correct boost velocity from the original unprimed frame to the comoving reference frame in which $\mathbf{S}' = 0$.¹²

Since \mathbf{V}_B is parallel to the energy flux vector \mathbf{S} , the energy flow velocity can also be written as $\mathbf{V}_B = V_B (\mathbf{S}/S)$ where the magnitude V_B is given by¹³

$$(V_B/c) = \frac{1}{(V_A/c)} \left\{ 1 - \sqrt{1 - (V_A/c)^2} \right\} \quad (17)$$

¹¹Note that $\mathbf{V}'_B \doteq \mathbf{V}_B$ as defined in Appendix I.1, together with eq.(12) and property (b) of the symbol \doteq in footnote 10, imply that $(\mathbf{V}'_B \cdot \mathbf{E}') = \mathbf{V}_B \cdot \gamma [\mathbf{E} + (\mathbf{V}_B/c) \times \mathbf{B}] = \gamma (\mathbf{V}_B \cdot \mathbf{E}) = 0$. Similarly, $(\mathbf{V}'_B \cdot \mathbf{B}') = 0$.

¹²Appendix V gives details of the comoving system for possible values of $(\mathbf{E} \cdot \mathbf{B})$ at a given event.

¹³The text just after eq.(2) proves that $0 \leq V_A \leq c$. As (V_A/c) increases from 0 to 1, eq.(17) shows that (V_B/c) increases monotonically from 0 to 1, with $V_B \leq V_A$ at every point. It follows that $0 \leq V_B \leq c$ also. Regions of the unprimed system where \mathcal{E} is nonzero but \mathbf{S} is zero have $V_A = 0$ and $V_B = 0$, and have no energy flow.

Eq.(17) can be inverted to give

$$(V_A/c) = \frac{2(V_B/c)}{1 + (V_B/c)^2} \quad (18)$$

which can be used to write the factor λ in eq.(15) as a function of the velocity Definition B

$$\lambda = \frac{1 + (V_B/c)^2}{2} \quad (19)$$

which shows $\lambda \leq 1$ and hence $V_B \leq V_A$.

Summary: This section uses the co-variant field transformation equations in eq.(10) to derive a boost velocity, \mathbf{V}_B , defined in eq.(16), that transforms from the unprimed system to a comoving primed system in which the energy flux vector $\mathbf{S}' = 0$. Then Appendix I.2 shows that \mathbf{V}_B is also the coordinate velocity relative to the unprimed system of an observer at rest in the comoving primed system. Since it is derived directly from the rules of special relativity, this velocity is well defined and relativistically correct. Also, it can be shown that this \mathbf{V}_B passes the *Einstein Addition Test*, as it must.

The observer at rest in the primed comoving system will observe the energy flux vector \mathbf{S}' to be zero. Thus if he holds an oriented area element $d\mathbf{a}'$ in any orientation he will find that the energy flux through that element to be $\mathbf{S}' \cdot d\mathbf{a}' = 0$. Hence the observer must be moving at the same velocity as the flow of energy, and its velocity will be the same as his velocity, \mathbf{V}_B .

The conclusion is that the well defined and relativistically correct coordinate velocity \mathbf{V}_B must be the correct velocity of the electromagnetic energy flow.

This conclusion, together with $\mathbf{V}_B \neq \mathbf{V}_A$ from eq.(16), also gives independent confirmation of the results of Section 3, that the correct definition of electromagnetic energy flow velocity is not the consensus value \mathbf{V}_A . It is important to note that this conclusion, along with all the results in Section 4, depend only on the assumption of the standard transformation laws of electromagnetic field in eq.(10), and not on any other assumptions.

Thus Section 4 provides convincing proof that \mathbf{V}_B is the relativistically correct electromagnetic energy flow velocity definition, and that the consensus value \mathbf{V}_A is not.

5 Caveats of Definition B

The caveats for Definition A are technical; they concern its violation of the transformation rules of special relativity. By contrast, the derivation of \mathbf{V}_B in Section 4 is completely consistent with special relativity throughout. Velocity \mathbf{V}_B is the relativistically correct velocity of an observer at rest in the primed comoving reference system, defined as a system in which the energy flux vector $\mathbf{S}' = 0$. It follows that \mathbf{V}_B is the relativistically valid energy flow velocity.

But one may question whether the condition $\mathbf{S}' = 0$ used in Section 4 truly implies that the comoving observer is mov-

ing at the same velocity as the underlying energy flow, as required for \mathbf{V}_B to be the correct energy flow velocity. For, as eq.(2) and eq.(16) directly prove, $\mathbf{V}_B \neq \mathbf{V}_A = \mathbf{S}/\mathcal{E}$ and hence $\mathbf{S} \neq \mathcal{E}\mathbf{V}_B$. It may seem that this inequality will block derivation of the Poynting theorem on which the meanings of \mathbf{S} and \mathcal{E} depend.

However, the equality of \mathbf{S} and $\mathcal{E}\mathbf{V}_B$ is *not* a necessary condition for the Poynting theorem. Derivation of the Poynting theorem is independent of the relation between \mathbf{S} and the product $\mathcal{E}\mathbf{V}_B$. The Poynting conservation of energy theorem derives from the divergence of the symmetric energy-momentum tensor $T^{\mu\nu}$ defined in eq.(5)

$$\partial_\mu T^{\mu\nu} = -f^\nu \quad \text{where} \quad f^\alpha = \frac{1}{c} F^\alpha_\beta J^\beta \quad (20)$$

is the Lorentz force density four-vector and $F^{\mu\nu}$ is the electromagnetic field tensor.¹⁴ The $\nu = 0$ component of the above manifestly covariant equation expands to

$$\frac{\partial \mathcal{E}}{\partial t} + \nabla \cdot \mathbf{S} = -\mathbf{E} \cdot \mathbf{J} \quad (21)$$

which is the Poynting work-energy theorem of electromagnetism. Since it is derived from the manifestly covariant pair of equations, eq.(20) the Poynting energy conservation formula eq.(21) is well defined and relativistically correct. And the meanings of \mathcal{E} and \mathbf{S} as energy density and energy-flux vector, respectively, are established by eq.(21). No further proof is required. The Poynting theorem and the meaning of \mathbf{S} as the energy flux vector are thus proved, regardless of the relation between \mathbf{S} and the product $\mathcal{E}\mathbf{V}_B$.

This proof that the Poynting theorem and the meaning of the energy flux vector \mathbf{S} are independently established corroborates and completes the argument at the end of Section 4, which depended on the meaning of \mathbf{S} . Thus we are driven to the conclusion that \mathbf{V}_B is indeed the well defined and relativistically correct velocity of the electromagnetic energy flow, and that \mathbf{V}_A is not.

6 An Exceptional Case

Although $\mathbf{V}_A \neq \mathbf{V}_B$ in general, there is an important exceptional case, which the theory here must approach as a limit. A plane, monochromatic, right/left circularly polarized light wave in vacuum with angular velocity ω and wave vector $\mathbf{k} = (\omega/c)\mathbf{e}_3$ has

$$\begin{aligned} \mathbf{E} &= E_0 \{ \mathbf{e}_1 \cos \phi \pm \mathbf{e}_2 \sin \phi \} \\ \mathbf{B} &= E_0 \{ \mp \mathbf{e}_1 \sin \phi + \mathbf{e}_2 \cos \phi \} \end{aligned} \quad (22)$$

where $\phi = (kz - \omega t)$ and $z = x^3$. This electromagnetic field has $\mathbf{E} \perp \mathbf{B}$ and $E = B = E_0 \neq 0$, which is the limiting case treated in item (c) of Appendix V. In this exceptional case,

¹⁴See Section 7.3 of Rindler[13].

velocity Definitions A and B coincide. As can be seen from eq.(2) and eq.(17) $V_B = c = V_A$.

As noted in Appendix V, and as also can be read from eq.(25), in this case \mathcal{E}' would be zero in the comoving system. But there is no comoving system with velocity magnitude equal to the speed of light. Observers are not permitted to ride on light waves. However, both definitions do agree that the flow speed of a light wave is the speed of light.

Setting $V_A = c$ and using $\mathbf{S} = c^2\mathbf{G}$, eq.(1) in this special case implies that

$$Gc = \mathcal{E} \quad (23)$$

Since wave solution eq.(22) defines a mode of the electromagnetic field whose second-quantization creates photons of definite vector momentum, eq.(23) can be considered a classical precursor of the relation $pc = e$ for the photon momentum and energy, a relation that requires the photon to be a massless particle.

7 The Energy-Momentum Tensor in a Comoving Frame

The derivation of velocity \mathbf{V}_B in Section 4 also allows the electromagnetic energy-momentum tensor in the comoving system to be derived from first principles. As noted in Section 4, the comoving energy-momentum tensor of a perfect fluid must simply be asserted rather than derived. But the electromagnetic energy-momentum tensor in a comoving system can be *derived*, and shown equal to a simple, diagonal form depending only on the energy density and one other parameter.

In the comoving (primed) coordinate system that was produced by the Lorentz boost \mathbf{V}_B , the energy-momentum tensor eq.(5) is represented by the tensor components $T'^{\alpha\beta}$ in which the $cG'_i = S'_i/c = 0$.

$$(T'^{\alpha\beta}) = \begin{pmatrix} \mathcal{E}' & 0 & 0 & 0 \\ 0 & M'_{11} & M'_{12} & M'_{13} \\ 0 & M'_{21} & M'_{22} & M'_{23} \\ 0 & M'_{31} & M'_{32} & M'_{33} \end{pmatrix} \quad (24)$$

where¹⁵

$$\mathcal{E}' = \frac{1}{2} (E'^2 + B'^2) = \mathcal{E} \frac{1 - (V_B/c)^2}{1 + (V_B/c)^2} \quad (25)$$

$$\text{and} \quad M'_{ij} = - (E'_i E'_j + B'_i B'_j) + \delta_{ij} \mathcal{E}'$$

We can now make another Lorentz transformation, an orthogonal spatial rotation at fixed time, to diagonalize the real, symmetric sub-matrix M'_{ij} in eq.(24).

The required spatial rotation can be defined as the product of two proper rotations. First, rotate the coordinate system to bring the \mathbf{e}'_3 axis into the $\mathbf{V}'_B \doteq \mathbf{V}_B$ direction.¹⁶ Denote this rotated system by tildes. Rotations do not change three-vectors, which are invariant objects under rotations. However,

¹⁵Eq.(25) is derived in Appendix IV.

¹⁶Note that item (c) of footnote 10 implies equal magnitudes $V'_B = V_B$.

rotations do change the *components* of three-vectors. Thus $\tilde{\mathbf{V}}_B = \mathbf{V}'_B$, $\tilde{\mathbf{E}} = \mathbf{E}'$, and $\tilde{\mathbf{B}} = \mathbf{B}'$, but in the tilde system $\tilde{\mathbf{V}}_B$ now has components $\tilde{V}_{B1} = \tilde{V}_{B2} = 0$ and $\tilde{V}_{B3} = V_B$. Then using footnote 11, we have $0 = (\mathbf{E}' \cdot \mathbf{V}'_B) = (\tilde{\mathbf{E}} \cdot \tilde{\mathbf{V}}_B) = V_B \tilde{E}_3$. Except in no-flow regions with \mathcal{E} nonzero but \mathbf{S} zero, the magnitude $V_B \neq 0$ and thus $\tilde{E}_3 = 0$. A similar argument proves that $\tilde{B}_3 = 0$. Thus the (33) component of the energy-momentum tensor when expressed in the tilde system is $\tilde{T}^{33} = -(\tilde{E}_3^2 + \tilde{B}_3^2) + \tilde{\mathcal{E}} = \tilde{\mathcal{E}}$. The tensor from eq.(24), when expressed in the tilde system, becomes

$$(\tilde{T}^{\alpha\beta}) = \begin{pmatrix} \tilde{\mathcal{E}} & 0 & 0 & 0 \\ 0 & \tilde{M}_{11} & \tilde{M}_{12} & 0 \\ 0 & \tilde{M}_{21} & \tilde{M}_{22} & 0 \\ 0 & 0 & 0 & \tilde{\mathcal{E}} \end{pmatrix} \quad (26)$$

where $\tilde{\mathcal{E}} = \mathcal{E}'$.

Since the invariant trace of the electrodynamic energy-momentum tensor vanishes,¹⁷ it follows from eq.(26) that

$$0 = \eta_{\alpha\beta} \tilde{T}^{\alpha\beta} = -\tilde{\mathcal{E}} + \tilde{M}_{11} + \tilde{M}_{22} + \tilde{\mathcal{E}} \quad (27)$$

and hence $\tilde{M}_{11} = -\tilde{M}_{22}$. Also, the symmetry of the energy-momentum tensor makes $\tilde{M}_{21} = \tilde{M}_{12}$. Thus

$$(\tilde{T}^{\alpha\beta}) = \begin{pmatrix} \tilde{\mathcal{E}} & 0 & 0 & 0 \\ 0 & -\tilde{\psi} & \tilde{\xi} & 0 \\ 0 & \tilde{\xi} & \tilde{\psi} & 0 \\ 0 & 0 & 0 & \tilde{\mathcal{E}} \end{pmatrix} \quad (28)$$

where $\tilde{\psi} = \tilde{M}_{22}$ and $\tilde{\xi} = \tilde{M}_{12}$.

A second proper rotation, this time about the $\tilde{\mathbf{e}}_3$ axis, produces the final coordinate system, denoted with double primes. After this rotation, $E''_3 = \tilde{E}_3 = 0$, $B''_3 = \tilde{B}_3 = 0$, and $\mathbf{V}''_B = \tilde{\mathbf{V}}_B$ has components $V''_{B1} = V''_{B2} = 0$ and $V''_{B3} = V_B$. The only effect of this second rotation is to diagonalize the 2x2 matrix $\begin{pmatrix} -\tilde{\psi} & \tilde{\xi} \\ \tilde{\xi} & \tilde{\psi} \end{pmatrix}$. The energy-momentum tensor then has its final diagonal form in the double-prime system

$$T''^{\alpha\beta} = \begin{pmatrix} \mathcal{E}'' & 0 & 0 & 0 \\ 0 & -a'' & 0 & 0 \\ 0 & 0 & a'' & 0 \\ 0 & 0 & 0 & \mathcal{E}'' \end{pmatrix} \quad (29)$$

where $\mathcal{E}'' = \tilde{\mathcal{E}} = \mathcal{E}'$. The parameter a'' has absolute value $|a''| = \{\tilde{\psi}^2 + \tilde{\xi}^2\}^{1/2}$ where $\pm \{\tilde{\psi}^2 + \tilde{\xi}^2\}^{1/2}$ are the two eigenvalues of the matrix $\begin{pmatrix} -\tilde{\psi} & \tilde{\xi} \\ \tilde{\xi} & \tilde{\psi} \end{pmatrix}$ that were calculated during the diagonalization process. The sign of a'' depends on the directions and relative magnitudes of the electric and magnetic fields.

¹⁷See Section 7.8 of Rindler [13]

The rotation that takes the system from the primed to the double-primed system is then the product of the first and second rotations. The various representations of the boost velocity used above are related by $\mathbf{V}''_B = V_B \mathbf{e}''_3 = \tilde{\mathbf{V}}_B = V_B \tilde{\mathbf{e}}_3 = \mathbf{V}'_B \triangleq \mathbf{V}_B$. It follows from item (c) of footnote 10 that all of these vectors have the same original magnitude V_B .

The energy-momentum tensor eq.(29) in the double-prime system is diagonal and in a canonical form, with two elements equal to $\mathcal{E}'' = \mathcal{E}'$ and two other elements equal to plus or minus the single parameter a'' .

The reduction of the electromagnetic energy-momentum tensor to the diagonal form in eq.(29) has important consequences for possible fluid-dynamic models of electromagnetic energy flow. For example, the perfect fluid model¹⁸ has a comoving energy-momentum tensor given by the diagonal matrix $(X'^{\alpha\beta}) = \text{diag}(\mathcal{E}', \pi', \pi', \pi')$ where \mathcal{E}' is an energy density and the π' are isotropic pressure terms, all of which are equal by definition. But, regardless of the value of parameters \mathcal{E}'' and a'' , there is no choice of \mathcal{E}' and π' for which the quadruplet of numbers $(\mathcal{E}', \pi', \pi', \pi')$ can match the quadruplet of numbers $(\mathcal{E}'', -a'', a'', \mathcal{E}'')$, other than the unphysical case when all of the numbers in both quadruplets are zero. Similarly, the so-called dust model¹⁹ has $(X'^{\alpha\beta}) = \text{diag}(\mathcal{E}', 0, 0, 0)$ which also cannot match the electromagnetic tensor.

Hence, the energy-momentum tensor of electrodynamics cannot be successfully modeled with either a perfect-fluid or a dust model.

8 Conclusion

The title of this paper asks whether electromagnetic field momentum is due to the flow of field energy. The answer has required careful examination of the velocity of energy flow.

First, consider velocity Definition A from Section 2. Dividing the definition $\mathbf{S} = \mathcal{E}\mathbf{V}_A$, from eq.(1) by c^2 and using $\mathbf{G} = \mathbf{S}/c^2$ and $\mathcal{M}_{\text{rel}} = \mathcal{E}/c^2$ gives

$$\mathbf{G} = \mathcal{M}_{\text{rel}} \mathbf{V}_A \quad (30)$$

which exhibits electromagnetic field momentum density \mathbf{G} as due to the flow of relativistic mass-energy \mathcal{M}_{rel} . But as shown in Section 3, $\mathbf{V}_A = \mathbf{S}/\mathcal{E}$ is not a relativistically correct velocity definition, and therefore must be rejected.

Definition B, on the other hand, is derived in Section 4 with complete adherence to the transformation rules of special relativity. Definition B is derived from the condition that the coordinate velocity of the flow of electromagnetic field energy at a given event is the velocity \mathbf{V}_B of a Lorentz boost that transforms the original reference system into a reference system in which the Poynting energy flux vector is zero at that event. But Definition B does not permit the explanation

¹⁸See Part I, Chapter 2, Section 10, eq.(2.10.1) *et seq* of Weinberg [20].

¹⁹Discussed in Section 12.2 of d'Inverno [4] and on page 301 *et seq.* of Rindler [13].

of field momentum density as due to moving relativistic mass-energy. Introducing $\mathbf{V}_B = \lambda \mathbf{V}_A$ from eq.(11) into eq.(30) and using eq.(19) gives

$$\mathbf{G} = \frac{M_{\text{rel}}}{\lambda} (\lambda \mathbf{V}_A) = \frac{M_{\text{rel}}}{\lambda} \mathbf{V}_B = \frac{2M_{\text{rel}} \mathbf{V}_B}{1 + (V_B/c)^2} \quad (31)$$

Flow of relativistic mass M_{rel} at velocity \mathbf{V}_B would produce a momentum density $M_{\text{rel}} \mathbf{V}_B = \lambda \mathbf{G}$ that has the same direction as \mathbf{G} but has a magnitude that is too small by the factor $\lambda \leq 1$ defined in eqs.(15, 19).

Note that this failure of the flow of relativistic mass M_{rel} to explain the field momentum density \mathbf{G} in the electromagnetic fields *must not be confused with the so-called hidden momentum* in the sources that is sometimes invoked to balance the field momentum and preserve momentum conservation globally.²⁰

The present paper is concerned only with a correct understanding of the electromagnetic *field contribution itself*, locally at every point of the electromagnetic field including those points with no source density. Encouraged by the arguments from the Feynman example noted in footnote 1 above, we accept that the vector $\mathbf{G} = \mathbf{S}/c^2$ correctly reproduces the local field momentum density at every point of the electromagnetic field. The question is the source of that local point-by-point field momentum density.

The conclusion of this paper can be now stated as the following proposition:

Proposition 2: *There is no relativistically correct definition of energy flow velocity that explains the electromagnetic field momentum density as due to the flow of field energy.*

Proof: *If an energy flow velocity \mathbf{v} obeys $\mathbf{G} = M_{\text{rel}} \mathbf{v}$, then multiplying by c^2 implies $\mathbf{S} = \mathcal{E} \mathbf{v}$ which in turn implies that $\mathbf{v} = \mathbf{S}/\mathcal{E} = \mathbf{V}_A$. But, according to Proposition 1 in Section 3, \mathbf{V}_A is not a relativistically correct velocity for electromagnetic energy flow. This \mathbf{v} is therefore not relativistically correct.*

The title question of this paper has a negative answer. When adherence to the strict transformation rules of special relativity is required, electromagnetic field momentum cannot be explained as due to the flow of field energy.

9 Afterword

Detailed studies of the energy and momentum carried by the electromagnetic field, such as the present paper, can be seen as searches for clues to a possible new physics underlying the Maxwell Equations. But, if we accept the conclusion at the end of Section 8, the attempt to model the Maxwell Equations at the level of energy flow and the energy-momentum tensor seems a failed program.

This failure calls into question the whole project of finding a deeper level behind the Maxwell Equations. A consensus exists that the electric and magnetic fields are not states

²⁰See Example 12.12 of Griffiths [8], and also McDonald [12] and Babson et al [1].

of anything else,²¹ but are either abstract mathematical aids, or themselves elements of reality to be taken as fundamental. In this view, the Maxwell Equations are already at the fundamental level, and attempts to derive them from some deeper reality are a futile revival of nineteenth century aether theories and, as Feynman says, "... produce nothing but errors."

But Maxwell himself looked for fluid models of his equations. Maxwell [11] explains the inverse square electric force law as a consequence of the spread of an incompressible fluid. And he later proposes (Maxwell [10]) a model of Faraday's magnetic field lines based on fluid vortices.²²

Perhaps, instead of taking the conclusion in Section 8 as a reason to abandon Maxwell's search, we should rather read a lesson from it: *Our attempt at a flow model may have failed because the attempt is taking place at the wrong level.* The electromagnetic energy-momentum tensor $T^{\alpha\beta}$ is quadratic in the fundamental electromagnetic fields E and B . It may be that any successful flow model of electrodynamics must operate at the linear, field level and not at the energy-momentum level.

For example, consider two monochromatic plane waves propagating in the $+\mathbf{e}_3$ direction, wave a with right circular polarization and wave b with left circular polarization.

$$\begin{aligned} \mathbf{E}_a &= E_0 \{ \mathbf{e}_1 \cos \phi + \mathbf{e}_2 \sin \phi \} \\ \mathbf{E}_b &= E_0 \{ \mathbf{e}_1 \cos \phi - \mathbf{e}_2 \sin \phi \} \end{aligned} \quad (32)$$

where $\phi = (kz - \omega t)$ and $z = x^3$, and the magnetic fields are the cross product of \mathbf{e}_3 with the given electric fields. The electromagnetic energy-momentum tensors $T_a^{\alpha\beta}$ and $T_b^{\alpha\beta}$ of these two waves are the same, with

$$(T_a^{\alpha\beta}) = (T_b^{\alpha\beta}) = \begin{pmatrix} E_0^2 & 0 & 0 & E_0^2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ E_0^2 & 0 & 0 & E_0^2 \end{pmatrix} \quad (33)$$

We now want to superpose these two situations a and b . The superposition of the two circularly polarized waves is a linearly polarized wave

$$\mathbf{E}_{a+b} = \mathbf{E}_a + \mathbf{E}_b = 2E_0 \mathbf{e}_1 \cos \phi \quad (34)$$

and the resulting energy-momentum tensor is

$$(T_{a+b}^{\alpha\beta}) = \begin{pmatrix} 4E_0^2 \cos^2 \phi & 0 & 0 & 4E_0^2 \cos^2 \phi \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 4E_0^2 \cos^2 \phi & 0 & 0 & 4E_0^2 \cos^2 \phi \end{pmatrix} \quad (35)$$

which is time varying at each fixed spatial point, passing through zero every π/ω seconds.

²¹See, for example, Section 4-5 of Feynman [6].

²²Falconer [5] surveys other early vortex models.

This example illustrates that representing the electromagnetic energy-momentum flow as the flow of a fluid at the quadratic energy-momentum level ignores the fact that electromagnetism is a linear theory with superposition. It is difficult to see how combining the two tensors in eq.(33) could result in the time-varying tensor of eq.(35). Electromagnetic fields do not superpose at the energy-momentum level. Therefore an attempt to model electromagnetism at that level is bound to fail. Such a model should be applied at the linear, field level of the E and B fields themselves.

But, in spite of the appeal and long history of Maxwell's quest, there are formidable hurdles facing any model at the field level, even using modern mathematical techniques. One such hurdle is that a complete model of the E and B fields would probably also need to include interaction with, and characterization of, the source fields ρ and \mathbf{J} . And it should include not only the effects of sources on fields but also the effects of fields on sources, the Lorentz force law.

Appendix I: Lorentz Boosts

Consider a Lorentz transformation from an "unprimed" coordinate system with coordinates $x = (x^0, x^1, x^2, x^3)$ to a "primed" coordinate system with coordinates $x' = (x'^0, x'^1, x'^2, x'^3)$ where $x^0 = ct$ and $x'^0 = ct'$. The most general proper, homogeneous Lorentz transformation from the unprimed to the primed systems can be written as a Lorentz boost times a rotation.²³

I.1: Definition of Lorentz Boost

A Lorentz boost transformation is parameterized by a boost velocity vector \mathbf{V} with components (V_1, V_2, V_3) and magnitude $V = (V_1^2 + V_2^2 + V_3^2)^{1/2}$. Using the Einstein summation convention, it is written as $x'^\alpha = \Lambda^\alpha_\beta x^\beta$ where $\Lambda^0_0 = \gamma$, $\Lambda^i_0 = \Lambda^i_0 = -\gamma V_i/c$, and $\Lambda^i_j = \delta_{ij} + (\gamma - 1)V_i V_j/V^2$. The δ_{ij} is the Kroeneker delta function, and $\gamma = (1 - V^2/c^2)^{-1/2}$.

The inverse boost $\underline{\Lambda}^\alpha_\beta$ is the same except for the substitution $V_i \rightarrow -V_i$. Thus the inverse boost vector is $(-\mathbf{V}')$ where $\mathbf{V}' \doteq \mathbf{V}$. (See footnote 10 for definition of the \doteq symbol.)

I.2: Meaning of the Boost Velocity \mathbf{V}

The velocity \mathbf{V} that parameterizes the Lorentz boost is also the coordinate velocity, as measured from the unprimed system, of any point that is at rest in the primed system. In this sense, the entire primed system is moving with velocity \mathbf{V} as observed from the unprimed system. Any observer at rest in the primed system is moving with that velocity \mathbf{V} relative to the unprimed system.

To see this, apply the inverse Lorentz boost to the differentials of a point at rest in the primed system, $dx'^i = 0$ for $i = 1, 2, 3$, but $dx'^0 > 0$. The result is $dx^0 = \gamma dx'^0$ and $dx^i = \gamma(V_i/c) dx'^0$. Thus $dx^i/dt = V_i$, as was asserted.

²³See Part I, Chapter 2, Section 1 of Weinberg [20].

Appendix II: Proof that Boost with \mathbf{V}_{qA} Makes $\mathbf{J}' = 0$

As applied to a four-vector $\mathbf{J} = J^0 \mathbf{e}_0 + \mathbf{J}$, with $J^0 = c\rho$ and $J^i = (\mathbf{J})_i$ the Lorentz boost transformation rule is $J'^\alpha = \Lambda^\alpha_\beta J^\beta$. Hence

$$\begin{aligned} J'^i &= \Lambda^i_0 J^0 + \Lambda^i_j J^j \\ &= -\gamma \frac{V_i}{c} J^0 + J^i + (\gamma - 1) \frac{V_i (V_j J^j)}{V^2} \end{aligned} \quad (36)$$

Replacing boost velocity ratio V_i/c by $(\mathbf{V}_{qA})_i/c = J^i/J^0$ in eq.(36) makes $J'^i = 0$, as asserted.

Appendix III: Proof that \mathbf{V}_{qA} is Consistent with the Einstein Velocity Addition Formula

Consider two reference frames, one denoted as unprimed and the other with asterisks. Let the orientation of the frames be as in the *Einstein Addition Test* in Section 3.

The four-vector charge flux in the asterisk system is then $\mathbf{J} = J^{*0} \mathbf{e}_0^* + J^{*1} \mathbf{e}_1^*$ where $J^{*0} = c\rho^*$ and $J^{*1} = (\mathbf{J}^*)_1$. From the standard inverse boost formula $J^\alpha = \underline{\Lambda}^\alpha_\beta J^{*\beta}$, the transform between the two frames is (suppressing the 2 and 3 components for simplicity)

$$\begin{pmatrix} J^0 \\ J^1 \end{pmatrix} = \gamma \begin{pmatrix} 1 & (V/c) \\ (V/c) & 1 \end{pmatrix} \begin{pmatrix} J^{*0} \\ J^{*1} \end{pmatrix} \quad (37)$$

The velocities V_{qA} and V_{qA}^* in the two frames are therefore related by

$$\begin{aligned} \frac{V_{qA}}{c} &= \frac{J^1}{J^0} = \frac{(V/c) J^{*0} + J^{*1}}{J^{*0} + (V/c) J^{*1}} \\ &= \frac{(V/c) + (J^{*1}/J^{*0})}{1 + (V/c)(J^{*1}/J^{*0})} = \frac{(V/c) + (V_{qA}^*/c)}{1 + (V V_{qA}^*/c^2)} \end{aligned} \quad (38)$$

which replicates the standard Einstein velocity addition formula, as asserted. Comparison of eq.(38) and eq.(4) shows that V_{qA} passes the *Einstein Addition Test*.

Appendix IV: Detailed Derivations of Eq.(13) and Eq.(25).

To derive eq.(13), we have eq.(2), eq.(11), eq.(12), and $(\mathbf{V} \cdot \mathbf{E}) = (\mathbf{V} \cdot \mathbf{B}) = 0$. Using eq.(12),

$$\mathbf{S}' = c(\mathbf{E}' \times \mathbf{B}') \doteq c\gamma_B^2 \{(\mathbf{E} \times \mathbf{B}) + \mathbf{f} + \mathbf{g}\}$$

where, omitting zero terms,

$$\begin{aligned} \mathbf{f} &= -\mathbf{E} \times \left(\frac{\mathbf{V}_B}{c} \times \mathbf{E} \right) + \left(\frac{\mathbf{V}_B}{c} \times \mathbf{B} \right) \times \mathbf{B} \\ &= -(E^2 + B^2) \frac{\mathbf{V}_B}{c} = -\lambda (E^2 + B^2) \frac{\mathbf{V}_A}{c} \\ &= -\lambda (E^2 + B^2) \frac{2(\mathbf{E} \times \mathbf{B})}{(E^2 + B^2)} = -2\lambda (\mathbf{E} \times \mathbf{B}) \end{aligned}$$

and, again omitting zero terms,

$$\begin{aligned}
\mathbf{g} &= -\left(\frac{\mathbf{V}_B}{c} \times \mathbf{B}\right) \times \left(\frac{\mathbf{V}_B}{c} \times \mathbf{E}\right) \\
&= -\frac{\mathbf{V}_B}{c} \left\{ \left(\frac{\mathbf{V}_B}{c} \times \mathbf{B}\right) \cdot \mathbf{E} \right\} = \frac{\mathbf{V}_B}{c} \left\{ \frac{\mathbf{V}_B}{c} \cdot (\mathbf{E} \times \mathbf{B}) \right\} \\
&= \lambda^2 \left\{ \frac{2(\mathbf{E} \times \mathbf{B})}{(E^2 + B^2)} \right\} \left\{ \frac{\mathbf{V}_A}{c} \cdot \left(\frac{E^2 + B^2}{2}\right) \frac{\mathbf{V}_A}{c} \right\} \\
&= \lambda^2 \left(\frac{\mathbf{V}_A}{c} \cdot \frac{\mathbf{V}_A}{c}\right) (\mathbf{E} \times \mathbf{B}) = \lambda^2 \left(\frac{V_A}{c}\right)^2 (\mathbf{E} \times \mathbf{B})
\end{aligned}$$

Collect terms and factor out $(\mathbf{E} \times \mathbf{B})$ to get

$$\mathbf{S}' = c(\mathbf{E}' \times \mathbf{B}') \doteq \gamma_B^2 c (\mathbf{E} \times \mathbf{B}) \left\{ \left(\frac{V_A}{c}\right)^2 \lambda^2 - 2\lambda + 1 \right\}$$

which is eq.(13).

To derive eq.(25) we have eq.(2), eq.(12), and $(\mathbf{V} \cdot \mathbf{E}) = (\mathbf{V} \cdot \mathbf{B}) = 0$. Using eq.(12), and property (b) of footnote 10,

$$\begin{aligned}
E'^2 &= \gamma_B^2 \left(\mathbf{E} + \frac{\mathbf{V}_B}{c} \times \mathbf{B} \right) \cdot \left(\mathbf{E} + \frac{\mathbf{V}_B}{c} \times \mathbf{B} \right) \\
&= \gamma_B^2 \left\{ E^2 + 2\mathbf{E} \cdot \left(\frac{\mathbf{V}_B}{c} \times \mathbf{B}\right) + \left(\frac{\mathbf{V}_B}{c} \times \mathbf{B}\right) \cdot \left(\frac{\mathbf{V}_B}{c} \times \mathbf{B}\right) \right\}
\end{aligned}$$

Omitting zero terms,

$$2\mathbf{E} \cdot \left(\frac{\mathbf{V}_B}{c} \times \mathbf{B}\right) = -2\frac{\mathbf{V}_B}{c} \cdot (\mathbf{E} \times \mathbf{B}) \quad \text{and}$$

$$\left(\frac{\mathbf{V}_B}{c} \times \mathbf{B}\right) \cdot \left(\frac{\mathbf{V}_B}{c} \times \mathbf{B}\right) = \left(\frac{\mathbf{V}_B}{c}\right) \cdot \left\{ \mathbf{B} \times \left(\frac{\mathbf{V}_B}{c} \times \mathbf{B}\right) \right\} = \left(\frac{V_B}{c}\right)^2 B^2$$

Thus

$$E'^2 = \gamma_B^2 \left\{ E^2 - 2\frac{\mathbf{V}_B}{c} \cdot (\mathbf{E} \times \mathbf{B}) + \left(\frac{V_B}{c}\right)^2 B^2 \right\}$$

Similarly,

$$B'^2 = \gamma_B^2 \left\{ B^2 - 2\frac{\mathbf{V}_B}{c} \cdot (\mathbf{E} \times \mathbf{B}) + \left(\frac{V_B}{c}\right)^2 E^2 \right\}$$

Combining, and using $(\mathbf{E} \times \mathbf{B}) = \left\{ 2\mathcal{E} / [1 + (V_B/c)^2] \right\} \frac{\mathbf{V}_B}{c}$ from eqs.(2, 11, and 19), where $\mathcal{E} = (E^2 + B^2)/2$, gives

$$\begin{aligned}
\mathcal{E}' &= \frac{1}{2}(E'^2 + B'^2) \\
&= \gamma_B^2 \left\{ \left[1 + (V_B/c)^2 \right] \frac{E^2 + B^2}{2} - 2\frac{\mathbf{V}_B}{c} \cdot (\mathbf{E} \times \mathbf{B}) \right\} \\
&= \frac{\gamma_B^2 \mathcal{E}}{1 + (V_B/c)^2} \left\{ \left[1 + (V_B/c)^2 \right]^2 - 4(V_B/c)^2 \right\} \\
&= \frac{\gamma_B^2 \mathcal{E}}{1 + (V_B/c)^2} \left[1 - (V_B/c)^2 \right]^2 = \mathcal{E} \frac{1 - (V_B/c)^2}{1 + (V_B/c)^2}
\end{aligned}$$

which is eq.(25).

Appendix V: Detail of the Comoving System

The comoving system is defined by $\mathbf{S}' = c(\mathbf{E}' \times \mathbf{B}') = 0$. Thus $|\mathbf{E}' \times \mathbf{B}'| = E' B' \sin \theta' = 0$ where θ' is the angle between \mathbf{E}' and \mathbf{B}' in the comoving system.

From eqs.(7.62 and 7.63) of Rindler [13], $(E'^2 - B'^2) = (E^2 - B^2)$ and $(\mathbf{E}' \cdot \mathbf{B}') = (\mathbf{E} \cdot \mathbf{B})$. It follows that:

(a) An event with $(\mathbf{E} \cdot \mathbf{B}) \neq 0$ has $E' B' \neq 0$ and therefore \mathbf{E}' and \mathbf{B}' must be either parallel or anti-parallel, $\theta' = 0$ or $\theta' = \pi$ at this event;

(b) An event with $0 = (\mathbf{E} \cdot \mathbf{B}) = (\mathbf{E}' \cdot \mathbf{B}') = E' B' \cos \theta'$ cannot have $E' B' \neq 0$ in the comoving system because that would require both $\cos \theta' = 0$ and $\sin \theta' = 0$. Thus $E' B' = 0$ and one of E' and B' must be zero. If $E > B$ then $E' > B'$ and hence $B' = 0$. If $E < B$ then $E' < B'$ and hence $E' = 0$;

(c) If both $0 = (\mathbf{E} \cdot \mathbf{B})$ and $E = B \neq 0$ at an event, then both $E' B' = 0$ and $E' = B'$, and therefore $E' = B' = 0$ and the fields and energy density \mathcal{E}' in the comoving system are zero. But eq.(2) and eq.(17) show that such an event also has $(V_A/c) = 1$ and hence $(V_B/c) = 1$ which is an unphysical value for a Lorentz boost velocity. The case $E = B \neq 0$ and $0 = \mathbf{E} \cdot \mathbf{B}$ therefore must be approached as a limit.

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