A Spherical Version of Feynman's Static Field Momentum Example

Oliver Davis Johns

San Francisco State University, Physics and Astronomy Department 1600 Holloway Avenue, San Francisco, CA 94133, USA Email: ojohns@metacosmos.org Web: https://www.metacosmos.org

Abstract

The Feynman demonstration that electromagnetic field momentum is real—even for static fields—can be made more pedagogically useful by simplifying its geometry. Instead of Feynman's disk with charged balls on its surface, this article uses the geometry of a hollow non-conducting sphere with uniform surface charge density. With only methods available in a typical upper-division electrodynamics course, the initial field angular momentum and the final mechanical angular momentum can then be calculated in closed form and shown to be equal.

This spherical geometry also provides a counterexample to the idea that electromagnetic field momentum is due to the classical flow of an inertial relativistic mass defined as the energy density divided by the square of the speed of light. The curved flow lines of an inertial field momentum would require a centripetal force to bend them, but no such force can be identified classically.

1 Introduction

The Feynman lectures¹ give an example of the reality of field momentum in a static electromagnetic field. A non-conducting, non-magnetic, horizontal disk is free to rotate about a vertical axis through its center. Several identical, charged balls are equally spaced around its circumference. In its center is an electromagnet. Due to cancellation, the net electric field is small on the surface of the disk. For distances from the axis of the disk larger than its radius, the electric and magnetic fields give a Poynting vector with a significant azimuthal component. The angular momentum density associated with this Poynting vector produces a nonzero total field angular momentum.

¹Feynman et al [3], Section 17-4, Section 27-6, and Figure 17-5.



Figure 1: The Feynman example. Charged balls are spaced evenly around the circumference of a disk. A small electromagnet is placed at the center.

The disk is at rest for times less than zero and hence has no mechanical angular momentum. If the electromagnet is programmed to turn itself off beginning at time zero, the resulting induced electric field exerts a torque on the charged spheres. After a long time, the magnetic field, and hence the Poynting vector and the field angular momentum, are zero. But the disk is now rotating as a result of the torque exerted on the spheres during the decay of the magnetic field. The initial field angular momentum has become final mechanical angular momentum, which shows that the initial field angular momentum must have been real.

The difficulty with this example is that it depends on qualitative arguments about the field strengths. Obviously, the magnetic field is stronger closer to the electromagnet. One must argue that the weakness of the electric field there more than compensates, so that the Poynting vector is large only in the region outside the radius of the disk where its azimuthal sense is correct. Such qualitative argumentation is not helpful for students who are, after all, not totally convinced that the phenomenon of field angular momentum is real.

Fortunately, it is not difficult to concoct a simplified, spherical example with fewer ambiguities. The methods used: multipole expansion of magnetic field and magnetic vector potential, Gauss's law for electric fields, retarded potentials for time-varying currents, the expression for the electric field when vector potential \mathbf{A} varies, are all part of a standard junior level Electricity and Magnetism course.² That level is also ideal for the introduction of examples such as this one.

2 The Spherical Feynman Example: Static-Field Angular Momentum

Suppose that a non-conducting, non-magnetic spherical shell of outer radius a is suspended by a vertical axis through its center (taken as the *z*-axis), about which it is free to rotate. The surface at radius a has a fixed, uniform surface

²For example, Wangsness [9] and Griffiths [4].

charge density σ_0 . Gauss's law then shows the electric field to be zero for r < a. At the center of the sphere is an electromagnet configured so that its total vector magnetic moment is $\eta_0 \hat{\mathbf{z}}$, parallel to the vertical axis. As in the original Feynman example, the magnetic moment η_0 is a positive constant for t < 0, and decreases to zero for t > 0. The sphere is at rest for negative time; there is nothing moving and hence no mechanical momentum for t < 0.



Figure 2: The simplified Feynman example. A non-conducting spherical shell of outer radius *a* is free to rotate about a fixed, non-conducting, and non-magnetic vertical shaft. At the center of the sphere, a small electromagnet is attached to the shaft.

All currents in the electromagnet are confined to be less than distance *b* from the sphere's center. Take b/a to be sufficiently small that, in the region on and outside radius *a* where the electric field is nonzero, the magnetic field can be approximated adequately by just the dipole term in a multipole expansion. The electric field is zero for r < a and is given by Gauss's law for r > a. Thus the static fields for r > a are³

$$\mathbf{E} = \frac{a^2 \sigma_0}{\varepsilon_0 r^2} \,\hat{\mathbf{r}} \qquad \qquad \mathbf{B} = \frac{\mu_0 \eta_0}{4\pi r^3} \left(2\cos\theta \,\hat{\mathbf{r}} + \sin\theta \,\hat{\theta} \right) \tag{2.1}$$

The Poynting vector is $\mathbf{S} = \mathbf{E} \times \mathbf{B}/\mu_0$ and the momentum density is $\mathbf{G} = \mathbf{S}/c^2$. The momentum density is thus zero for r < a and

$$\mathbf{G} = \frac{\mu_0 a^2 \sigma_0 \eta_0 \sin \theta}{4\pi r^5} \,\hat{\boldsymbol{\phi}} \tag{2.2}$$

³This paper uses MKSA units in vacuum with the speed of light $(\varepsilon_0\mu_0)^{-1/2}$ denoted by *c*. Spherical polar coordinates *r*, θ , ϕ are used with unit vectors $\hat{\mathbf{r}}$, $\hat{\theta}$, $\hat{\phi}$, the *z*-axis vertical along the shaft, and the origin at the center of the sphere. The *x* and *y*-axes are not shown in Figure 2.

for r > a. The field angular momentum density for r > a is therefore

$$\mathbf{r} \times \mathbf{G} = -\frac{\mu_0 a^2 \sigma_0 \eta_0 \sin \theta}{4\pi r^4} \,\hat{\boldsymbol{\theta}} \tag{2.3}$$

The total initial static field angular momentum is then

$$\mathbf{L}_0 = \int_a^\infty dr \, r^2 \int_0^\pi d\theta \sin\theta \int_0^{2\pi} d\phi \, \mathbf{r} \times \mathbf{G} = \frac{2\mu_0 a\sigma_0 \eta_0}{3} \, \hat{\mathbf{z}}$$
(2.4)

3 The Spherical Feynman Example: Torques and their Impulse

At time zero, the current in the electromagnet begins to decrease slowly. Take its magnetic moment for $t \ge 0$ to be $\eta(t) = \eta_0 e^{-t/\tau}$. Taking $\tau \gg a/c$ so that time delay effects are negligible, there is an instantaneous magnetic vector potential in the vicinity of r = a given, again, by just the dipole term in the multipole expansion

$$\mathbf{A}(\mathbf{r},t) = \frac{\mu_0 \eta(t) \sin \theta}{4\pi r^2} \hat{\phi}$$
(3.1)

From this vector potential and the surface charge density σ_0 , we use $\mathbf{E} = -\nabla \Phi_{\text{elec}} - \partial \mathbf{A} / \partial t$ to derive the instantaneous electric field at radius r = a

$$\mathbf{E}(a,\theta,\phi,t) = \frac{\sigma_0}{\varepsilon_0}\,\hat{\mathbf{r}} - \frac{\mu_0 \sin\theta}{4\pi a^2}\,\frac{d\eta}{dt}\,\hat{\phi}$$
(3.2)

The total instantaneous torque of this electric field on the surface charge density is then

$$\mathbf{N} = a^2 \int_0^{\pi} d\theta \sin \theta \int_0^{2\pi} d\phi \ a \ \hat{\mathbf{r}} \times \sigma_0 \mathbf{E}(a, \theta, \phi, t) = -\frac{2\mu_0 a \sigma_0}{3} \frac{d\eta}{dt} \, \hat{\mathbf{z}}$$
(3.3)

The total impulse of this torque is the final, mechanical angular momentum

$$\mathbf{L}_{\rm f} = \int_0^\infty dt \, \mathbf{N} = \frac{2\mu_0 a \sigma_0 \eta_0}{3} \, \hat{\mathbf{z}}$$
(3.4)

The initial field angular momentum eqn (2.4) equals the final mechanical angular momentum eqn (3.4). The simplifying assumptions made are internally consistent. We explicitly assumed $a \gg b$, and $\tau \gg a/c$. An implicit assumption, that the magnetic field produced by the rotating sphere produces negligible residual field angular momentum, can be ensured by using a sphere with very large moment of inertia so that its angular velocity remains small.

4 The Quantitative Feynman Argument

The principle of angular momentum conservation requires that the final mechanical angular momentum $\mathbf{L}_{\rm f}$ of the rotating sphere at $\tau \gg a/c$ must have already been present for t < 0. Since there was no motion for t < 0, that angular momentum can only have been stored in the initial static field configuration.

The quantitative agreement $\mathbf{L}_{f} = \mathbf{L}_{0}$ between eqn (3.4) and eqn (2.4) is convincing proof that the final mechanical angular momentum was indeed stored initially in the static electromagnetic field, and that the static field angular momentum \mathbf{L}_{0} must therefore have been physically real. It follows that the linear momentum **G** in eqn (2.2) must also be physically real.⁴

5 Other Geometries

Previous attempts to verify the Feynman example quantitatively have used geometries not closely related to the Feynman disk with balls on its surface.

Romer [7] uses a very long, hollow, fixed solenoid containing a fixed, coaxial conducting rod. At t = 0, a spot of radioactive material on the rod emits a particle of positive charge which pierces the solenoid and escapes to infinity. As it escapes, this particle is given some angular momentum by the magnetic field of the solenoid. After the particle escapes to infinity, there is field angular momentum from the combined electromagnetic fields of the solenoid and the now-charged rod. This field angular momentum is shown to be equal in magnitude and opposite in direction to the angular momentum that was conveyed to the escaping particle. However, this example, and a second one in Romer's paper, do not directly show field angular momentum becoming mechanical angular momentum.

Lombardi [6] provides a formal and geometry-independent proof that angular momentum transferred to charges by a changing magnetic field must equal the change in field angular momentum. However this paper does not calculate specific values in a pedagogically useful way.

Corinaldesi [2] and Boos [1] also use a long, fixed, hollow solenoid. It contains two hollow, coaxial, nonconducting cylinders that have opposite surface charge densities and are free to rotate about their common axis. This geometry is like the Feynman example inside-out. The electromagnet is a solenoid on the outside and the moving parts are inside. These papers then turn off the magnetic field slowly and calculate the total impulse of the torque on the inner cylinders, showing it to equal the initial field angular momentum. However, this geometry can only compare values per unit length of the cylinder, and also has problems with fringing at the cylinders' top and bottom.

⁴The **G** in eqn (2.2) has no component parallel to \hat{r} and hence the whole of momentum density **G** contributes to the angular momentum density **r** × **G** in eqn (2.3).

6 Is Field Momentum Due to Classical Flow of Inertial Mass?

The motivation for the spherical Feynman geometry in this paper is principally pedagogical—to show students the reality of static field momentum. However, a curious student will then wonder how a dynamic thing like momentum can possibly be contained in a static field. The spherical Feynman geometry is an ideal testbed to consider some answers to that question.

An entirely classical explanation of static field momentum has been proposed,⁵ based on the idea that electromagnetic energy $\mathcal{E} = (\varepsilon_0 E^2 + B^2/\mu_0)/2$ contains an inertial relativistic mass $\mathcal{M} = \mathcal{E}/c^2$. Then a conventional formula defining the velocity **v** of energy flow

$$\mathbf{S} = \mathcal{E} \mathbf{v} \tag{6.1}$$

can be divided by c^2 to yield a relation between momentum **G** = **S**/ c^2 and relativistic mass

$$\mathbf{G} = \frac{\mathbf{S}}{c^2} = \frac{\mathcal{E}}{c^2} \mathbf{v} = \mathcal{M} \mathbf{v}$$
(6.2)

thus exhibiting static field momentum as due to the flow of inertial relativistic mass.

But, in spite of its plausibility, the energy-flow velocity v defined in eqn (6.1) and used in eqn (6.2) is inconsistent with the transformation rules of special relativity. There is no relativistically correct velocity v that satisfies eqn (6.1) and eqn (6.2).⁶ The field momentum **G** cannot be explained classically as due to the flow of an inertial relativistic mass $\mathcal{M} = \mathcal{E}/c^2$.

The spherical Feynman example also permits a more direct argument against a classical explanation based on an inertial mass \mathcal{M} . Evaluating eqn (2.2) in the equatorial plane $\theta = \pi/2$ at r = 2a shows the field lines of momentum $\mathbf{G}(r, \theta, \phi)$ there as forming a circle of radius 2a

$$\mathbf{G}(2a, \pi/2, \phi) = \frac{\mu_0 \sigma \eta_0}{32\pi a^3} \,\hat{\phi}$$
(6.3)

If there is some velocity v and inertial mass M such that Mv = G, then a centripetal force density

$$\mathbf{F} = -\frac{\mathcal{M}v^2}{2a}\hat{r} \tag{6.4}$$

is required to hold that mass in its circular orbit.

But there is no such force density \mathbf{F} in classical electromagnetism. Charges and currents act on fields, and fields act on charges and currents, but fields do not act on fields. Classical electrodynamics is a linear theory.

The spherical Feynman example is thus an ideal vehicle for demonstrating to students both that static field momentum is real, and also that it cannot be explained as the classical flow of some kind of inertial mass.

⁵See Sebens [8].

⁶See Propositions 1 and 2 of the 2023 revision of Johns [5].

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