Validity of the Einstein Hole Argument*

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Abstract

Arguing from his "hole" thought experiment, Einstein became convinced that, in cases in which the energy-momentum-tensor source vanishes in a spacetime hole, a solution to his general relativistic field equation cannot be uniquely determined by that source. After reviewing the definition of active diffeomorphisms, this paper uses them to outline a mathematical proof of Einstein's result. The relativistic field equation is shown to have multiple solutions, just as Einstein thought. But these multiple solutions can be distinguished by the different physical meaning that each metric solution attaches to the local coordinates used to write it. Thus the hole argument, while formally correct, does not prohibit the subsequent rejection of spurious solutions and the selection of a physically unique metric. This conclusion is illustrated using the Schwarzschild metric. It is suggested that the Einstein hole argument therefore cannot be used to argue against substantivalism.

1 Introduction

Einstein's "hole" thought experiment convinced him that specification of the energy-momentum-tensor source would not determine a unique solution to his general relativistic field equation.[1]

Einstein's own description of the argument was brief and lacking in detail. He first refers to the required transformations as what translates as *coordinate* transformations, and later as *point* transformations.[2] Stachel (1986) has interpreted this latter phrase as referring to what he calls "active diffeomorphisms."[3] In an attempt to avoid misunderstandings about notation and definitions, Section 2 makes some preliminary remarks and Section 3 uses basic differential geometry to define Stachel's term "active diffeomorphism" and its companion term "passive diffeomorphism."

Einstein posited a specific experimental situation in which a "hole" region H in spacetime is devoid of energy-momentum-tensor sources ($T_{\mu\nu}(x) \equiv 0$ for $x \in H$), with this hole surrounded by a source region S in which the energy-momentum tensor could be nonzero.[4] He argued that an active diffeomorphism that acted as the identity in the S region, but was not an identity in the hole, would modify the metric field in the hole without modifying any of the sources, either inside or outside the hole. He concluded that the energy-momentum sources cannot determine the metric field in the hole uniquely. Section 4 outlines a proof of Einstein's conclusion.

But the existence of a mathematical proof that Einstein's field equation has multiple solutions leads to the question of the physical meaning of these multiple solutions.[5] This issue is addressed in Section 5, which discusses the difficulties introduced into differential geometry by Einstein's disruptive idea of a Riemannian metric that is not known until after a differential equation for it is solved. Before the field equation is solved, since there is not yet a defined metric, the local coordinates are just m-tuples of real numbers that have no definite relation to anything physical like relativistic interval. After the field equation is solved, each of the multiple solutions produced by the hole argument is then a distinct metric that attaches its own distinct physical meaning to the local coordinates that were used to write it. It may thus be possible to select among the multiplicity of mathematical solutions of Section 4 a unique one that assigns to its local coordinates the physical meaning needed to model the symmetries of the experimental situation under study, rejecting the other metric solutions as spurious. Thus the hole argument in Section 4 fails to prove that Einstein's field equation must necessarily have multiple non-spurious solutions.

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[1] See Chapter 5 of Torretti (1996) and Chapter 5 of Stachel (2002) for the history of Einstein's quest for the equations of general relativity.

[2] See Section 5.6 of Torretti (1996).

[3] Einstein's term would be "point diffeomorphism." I use the terms "active diffeomorphism" and "point diffeomorphism" as exact synonyms.

[4] Einstein and Grossmann (1913). See also paraphrase by Torretti (1996), p. 163.

[5] By "physical meaning" (sometimes shortened to just "meaning") of a solution I mean a set of defined relations between the local coordinates used to write it and something like length or relativistic interval, such as is defined by a Riemannian metric.

Sections 6 and 7 illustrate these ideas using the Schwarzschild solution for a spherically symmetric source mass. In this case, a unique solution is found, thus providing a counterexample to the proposition that the Einstein field can never have a unique solution.

2 Preliminary Remarks

A few preliminary remarks may be helpful. First, in discussing the uniqueness of solutions to generally covariant differential equations, it is necessary to remember that any solution must be expressed in some system of local coordinates. A solution written in one coordinate system can, by a diffeomorphic change of local coordinates (passive diffeomorphism as defined in Section 3.1), always be transformed into the same solution expressed in some other coordinate system. (Think of converting from Cartesian to spherical polar coordinates in Euclidean three-space.) But the existence of these two expressions in the two coordinate systems is not what is meant when one speaks of non-uniqueness of solution. These are not *different* solutions, but only the *same* solution expressed in two different systems of local coordinates. To say that a generally covariant differential equation has a second solution and therefore is non-unique means a second solution that is different from the original one when *both* solutions are expressed in the *same* local coordinate system. This is the sense in which Einstein used the term "unique," and also the sense in which it is used in this paper.

Second, it is necessary to realize that there are at least two distinct and non-equivalent definitions of the hole argument extant in the literature. The first is that due to Einstein outlined above. There is no evidence that Einstein ever intended his hole argument to apply to generally covariant differential equations other than his own general relativistic field equation. Also, as will be shown in Section 4, Einstein's version depends essentially on his assumption of a particular experimental situation in which the energy momentum tensor term $T_{\mu\nu}(x)$ in his differential equation vanishes identically in the region he calls the "hole."

On the other hand, the revision of the hole argument by Earman and Norton[6] is asserted by them to include "...Newtonian spacetime theories with all, one, or none of gravitation and electrodynamics; and special and general relativity, with and without electrodynamics."[7] Also their presentation of the hole argument does not require that a source term must vanish in the hole region. They assert that Einstein's presentation is, only "...a specialized form..." of their generalized hole argument.[8]

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[6]Earman and Norton (1987); Norton (2011)

[7]However, the generally covariant Poisson equation for the electrostatic potential in three dimensions, when applied to a spherically symmetric source with the generally covariant boundary condition that the potential vanish at infinity, is well known to have a unique solution, thus providing a counterexample to Earman and Norton's assertion that their hole argument applies also to such differential equations.

[8]Earman and Norton, op.cit., p. 523. The Earman-Norton version explores the consequences of a Leibnizian interpretation of active diffeomorphisms. It makes no direct reference to the details of

This existence of two distinct hole arguments has confused the subject, with some refutations of what their authors take to be *the* hole argument apparently applying only to the Earman-Norton version.[9] This paper will not derive or defend the Earman-Norton version.

The third preliminary remark concerns style. It has become common to discuss the hole argument in abstract mathematical language.[10] But the subtlety of Einstein's argument is revealed only when one uses coordinates to study it. Fortunately, although invariant language is the norm today, arguments using coordinates are not therefore invalid. They may seem crude, but they are still true.

There is an analogy here to computer programming languages. High-level languages such as Python or C++ are elegant and succinct, but every programmer knows that there are some problems that require low-level machine assembly language to solve. In this paper, I discuss the Einstein hole argument using high-level invariant language—and assembly language when required.

3 Active and Passive Diffeomorphisms

This section outlines the definition of the term "active diffeomorphisms" and gives a method for generating them.

The distinction between active and passive diffeomorphisms is borrowed from the transformation theory of classical vector calculus.[11] Suppose that a three-dimensional Euclidean coordinate system containing a velocity $\mathbf{V}$ and another field $\mathbf{B}$ at point $\mathbf{r}$ is rotated by angle α about the z-axis as shown on the left side of Figure 3.1. Suppose that before rotation, the components of the vectors are $\mathbf{r} : (x^1, x^2, x^3)$, $\mathbf{V} : (V^1, V^2, V^3)$, and $\mathbf{B} : (B^1, B^2, B^3)$. After the rotation the vectors are unchanged, but their components become $\mathbf{r} : (x'^1, x'^2, x'^3)$, $\mathbf{V} : (V'^1, V'^2, V'^3)$, and $\mathbf{B} : (B'^1, B'^2, B'^3)$ where

$$x'^1 = x^1 \cos\alpha - x^2 \sin\alpha \qquad x'^2 = x^1 \sin\alpha + x^2 \cos\alpha \qquad x'^3 = x^3 \tag{3.1}$$

$$V'^1 = V^1 \cos\alpha - V^2 \sin\alpha \qquad V'^2 = V^1 \sin\alpha + V^2 \cos\alpha \qquad V'^3 = V^3 \tag{3.2}$$

with similar expressions for the components of $\mathbf{B}$. The observer, here represented by the coordinate system, rotates by angle α but the physical world being observed, here represented by the vectors, does not rotate. This is called a passive transformation since the world is not changed, just the view of the observer.

An active transformation rotates the physical world by angle α about the z-axis while keeping the observer fixed, as shown on the right side of Figure 3.1.

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the Einstein field equation, which details are the main focus of the Einstein version studied in the present paper.

[9] See for example the recent articles: Weatherall (2018); Schulman (2016)

[10] For example, the use of category theory in Iftime and Stachel (2006).

[11] For example, in Chapter 8 of Johns (2011) active transformations are used initially and passive transformations are introduced in Section 8.30.
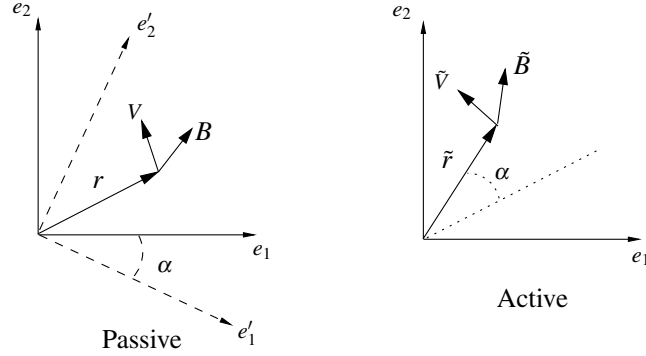
Figure 3.1: Active and passive transformations. The passive one rotates the co-ordinate system (observer) but leaves the vectors (physical world) unchanged. The active transformation rotates the vectors but leaves the coordinate system unchanged.

The observer's coordinate system is not changed, but the vectors are changed to new vectors $\tilde{\mathbf{r}}$, $\tilde{\mathbf{V}}$, and $\tilde{\mathbf{B}}$, with components (expressed in the unchanged original coordinate system)

$$\tilde{x}^1 = x^1 \cos\alpha - x^2 \sin\alpha \qquad \tilde{x}^2 = x^1 \sin\alpha + x^2 \cos\alpha \qquad \tilde{x}^3 = x^3 \tag{3.3}$$

$$\tilde{V}^1 = V^1 \cos\alpha - V^2 \sin\alpha \qquad \tilde{V}^2 = V^1 \sin\alpha + V^2 \cos\alpha \qquad \tilde{V}^3 = V^3 \tag{3.4}$$

with similar expressions for $\tilde{\mathbf{B}}$. This is called an active transformation since the observed world is changed but the observer is kept fixed.

3.1 Passive Diffeomorphism

The obvious differential geometric analog of the classical passive transformation of vector components in eqn (3.1) is the diffeomorphic change of local coordinates on a smooth manifold.[12]

Let a manifold $\mathcal{M}$ of dimension m have two overlapping charts of local coordinates (ψ, U) and (ψ', U') where U and U' are open sets in $\mathcal{M}$ with $U \cap U' \neq 0$, and ψ, ψ' are homeomorphisms from U, U' to local coordinates $x = (x^1, \dots, x^m)$ and $x' = (x'^1, \dots, x'^m)$, respectively, in $\mathbb{R}^m$. If the function $x' = \gamma(x)$, where $\gamma = \psi' \circ \psi^{-1}$, and its inverse $x = \gamma^{-1}(x')$ are both continuously differentiable to arbitrary order for all such overlapping open sets U, U', the manifold $\mathcal{M}$ is a smooth manifold and $\gamma : x \to x'$ is a diffeomorphic change of local coordinates. A point $p \in \mathcal{M}$ is represented either by local coordinates $x = \psi(p)$ or $x' = \psi'(p)$. Such "diffeomorphic changes of local coordinates" are referred to in this paper

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[12]See Lee (2013, 2010), texts I take to be the canonical references for modern, invariant differential geometry.

as "passive diffeomorphisms."[13]

Smooth functions $f : \mathcal{M} \to \mathbb{R}$ mapping points p on the smooth manifold to real numbers $f(p)$ are represented in unprimed and primed local coordinates by $F = f \circ \psi^{-1}$ and $F' = f \circ \psi'^{-1}$ so that[14]

$$F(x) = f(p) = F'(x') \tag{3.5}$$

One must distinguish between manifold objects[15] like $f(p)$ and local coordinate objects like $F(x)$. In physical theories, manifold objects can be taken as real while local coordinate objects only *represent* the underlying manifold ones in various local coordinate systems.

A tangent vector field $\mathbf{V}(p)$ is a manifold object in the tangent bundle of $\mathcal{M}$, a member of the tangent space over manifold point p. Its action is represented in operator notation; it maps smooth functions $f(p)$ to invariant real numbers denoted as $\mathbf{V}(p)f(p)$. It is represented in the unprimed and primed charts by

$$V(x) = \sum_{j=1}^{m} V^j(x)E_j \qquad\qquad V'(x') = \sum_{i=1}^{m} V'^i(x')E_i' \tag{3.6}$$

respectively, where $E_j = \partial/\partial x^j$ and $E_i' = \partial/\partial x'^i$ are local coordinate representations of basis vectors in the two charts. Then

$$V(x)F(x) = \mathbf{V}(p)f(p) = V'(x')F'(x') \tag{3.7}$$

and the components are related by the rule

$$V'^i(x') = \sum_{j=1}^{m} \frac{\partial x'^i}{\partial x^j} V^j(x) \tag{3.8}$$

A covariant tensor field of rank k is a manifold object $\mathbf{g}(p)$ that maps an ordered set of tangent vector fields to an invariant real number denoted

$$\mathbf{g}(p)\{\mathbf{V}_1(p), \ldots, \mathbf{V}_k(p)\} \tag{3.9}$$

If the manifold $\mathcal{M}$ is Riemannian[16] with second rank, covariant metric tensor field $\mathbf{g}(p)$, denoted $(\mathcal{M}, \mathbf{g})$, the invariant inner product of two tangent vector fields is defined as

$$\big\langle \mathbf{V}(p), \mathbf{W}(p) \big\rangle = \mathbf{g}(p)\{\mathbf{V}(p), \mathbf{W}(p)\} \tag{3.10}$$

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[13]The term "gauge transformations of differential geometry" is also sometimes used in the literature.

[14]In the literature, function $F(x)$ is often written $f(x)$. One is supposed to read from the variable, x rather than p, that F is intended. The condition that f be a smooth function is that local coordinate function $F(x)$ must be continuously differentiable to arbitrary order.

[15]The terms "manifold object" and "invariant object" are used as synonyms in this paper. Manifold objects like $f(p)$ are invariant under changes of local coordinates.

[16]In this paper, Riemannian always is intended to include Semi-Riemannian.

The metric tensor field is represented in the unprimed and primed charts by components $g_{ij}(x)$ and $g'_{kl}(x')$, respectively. The inner product is then

$$\sum_{i,j=1}^{m} g_{ij}(x)V^i(x)W^j(x) = \left\langle \mathbf{V}(p), \mathbf{W}(p) \right\rangle = \sum_{k,l=1}^{m} g'_{kl}(x')V'^k(x')W'^l(x') \qquad (3.11)$$

and the local components of $\mathbf{g}(p)$ transform as

$$g_{ij}(x) = \sum_{k,l=1}^{m} g'_{kl}(x')\frac{\partial x'^k}{\partial x^i}\frac{\partial x'^l}{\partial x^j} \qquad (3.12)$$

3.2   Active Diffeomorphism

In Section 3.1, the differential geometric analog of classical vector passive trans-formations was easily available; one simply identified "passive diffeomorphisms" with universally accepted definition of "diffeomorphic change of local coordi-nates." But the differential geometric analog of the active transformation of clas-sical vectors in eqn (3.3), to be called an "active diffeomorphism" here, is less well established and requires some definition. Some texts on differential geom-etry for the general relativity community, e.g., Carroll (2016); Wald (1984), dis-cuss active diffeomorphisms peripherally, but other standard references on dif-ferential geometry for the pure mathematics and high-energy physics commu-nities, e.g., Frankel (2004); Lee (1997, 2013); O'Neill (1983); Taubes (2011), do not even contain the phrase. However, they do contain a construction that can be tailored to our purposes, the differentiable mapping $\phi : \mathcal{M} \to \mathcal{N}$ be-tween two manifolds $\mathcal{M}$ and $\mathcal{N}$ of dimension m and n, respectively, where in general the dimensions are different, $m \neq n$, and the mapping need not be a homeomorphism (a continuous mapping with a continuous inverse).[17]

Here we consider the restricted case in which $m = n$ and ϕ is an active diffeomorphism. Thus, if (U, ψ) and $(\tilde{U}, \tilde{\psi})$ are charts of local coordinates x, $\tilde{x}$ on $\mathcal{M}$ and $\mathcal{N}$, respectively, we assume that both $\tilde{x} = \theta(x)$, where $\theta = \tilde{\psi} \circ \phi \circ \psi^{-1}$, and its inverse $x = \theta^{-1}(\tilde{x})$ exist and are continuously differentiable to arbitrary order.[18]

Let p and $\tilde{p} = \phi(p)$ be points in $\mathcal{M}$ and $\mathcal{N}$, respectively, and let $\tilde{f}(\tilde{p})$ be a smooth function $\tilde{f} : \mathcal{N} \to \mathbb{R}$. Then there is a smooth function $f(p)$ with $f : \mathcal{M} \to \mathbb{R}$ defined by $f = \tilde{f} \circ \phi$. This f is called the *pull-back* of $\tilde{f}$ and is denoted $f = \phi^* \tilde{f}$. Since ϕ is assumed here to have an inverse, the $\tilde{f}$ can also be written as what is called the *push-forward* of f, denoted $\tilde{f} = \phi_* f$. No matter how denoted, the relation is

$$\tilde{f}(\tilde{p}) = f(p) \qquad (3.13)$$

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[17]See Chapters 2 and 3 of Lee (2013).

[18]It will be assumed uncritically here that the domains of the homeomorphisms ψ, ψ', and $\tilde{\psi}$ which define the local coordinates comprise the whole of their respective manifolds. If multiple domains are required in a particular case, it is assumed that they can be patched together by standard techniques.

This relation can also be written in local coordinates. If $\tilde{F}(\tilde{x})$ is a smooth function defined in terms of local coordinates on $\mathcal{N}$, then there is a smooth function $F(x)$, where $F = \tilde{F} \circ \theta$, similarly defined on $\mathcal{M}$. This F is called a pull-back of $\tilde{F}$ and is denoted $F = \phi^* \tilde{F}$. Since we are assuming ϕ and hence θ to have an inverse, we can also refer to $\tilde{F}$ as what is called a push-forward of F denoted $\tilde{F} = \phi_* F$. In either case, the relation is

$$F(x) = \tilde{F}(\tilde{x}) \tag{3.14}$$

which shows that $\tilde{F}(\tilde{x})$ at a point $\tilde{x}$ has the same value as function $F(x)$ has at point x.

In general, since ϕ and θ are assumed to be diffeomorphic here, and all transformations therefore possess inverses, both pull-back and push-forward of functions, tangent vectors, and general tensor fields are well defined.

Tangent vector fields can also be pulled back or pushed forward. Let $\mathbf{V}(p)$ and $\tilde{\mathbf{V}}(\tilde{p})$ be manifold objects on $\mathcal{M}$ and $\mathcal{N}$, respectively. Then $\mathbf{V} = \phi^* \tilde{\mathbf{V}}$, or equivalently $\tilde{\mathbf{V}} = \phi_* \mathbf{V}$, is defined by

$$\mathbf{V}(p)f(p) = \tilde{\mathbf{V}}(\tilde{p})\tilde{f}(\tilde{p}) \tag{3.15}$$

In terms of local coordinates with $\tilde{x} = \theta(x)$, this is

$$V(x)F(x) = \tilde{V}(\tilde{x})\tilde{F}(\tilde{x}) \tag{3.16}$$

and the local coordinate transformation, here written as a push-forward, is

$$\tilde{V}^i(\tilde{x}) = \sum_{j=1}^{m} \frac{\partial \tilde{x}^i}{\partial x^j} V^j(x) \tag{3.17}$$

In mappings between Riemannian manifolds $\phi : (\mathcal{M}, \mathbf{g}) \to (\mathcal{N}, \mathbf{h})$, the metric tensor also can be equivalently pulled back $\mathbf{g} = \phi^* \tilde{\mathbf{g}}$ or pushed forward $\tilde{\mathbf{g}} = \phi_* \mathbf{g}$. The definition is

$$\mathbf{g}(p)\{\mathbf{V}(p), \mathbf{W}(p)\} = \tilde{\mathbf{g}}(\tilde{p})\{\tilde{\mathbf{V}}(\tilde{p}), \tilde{\mathbf{W}}(\tilde{p})\} \tag{3.18}$$

for any general pair of tangent vectors. The component relation, here expressed as a pull-back, is

$$g_{ij}(x) = \sum_{k,l=1}^{m} \tilde{g}_{kl}(\tilde{x}) \frac{\partial \tilde{x}^k}{\partial x^i} \frac{\partial \tilde{x}^l}{\partial x^j} \tag{3.19}$$

The pushed forward metric $\tilde{\mathbf{g}} = \phi_* \mathbf{g}$ may or may not be the same as a pre-existing metric $\mathbf{h}$ of manifold $\mathcal{N}$. The case in which ϕ is diffeomorphic (as is assumed here), and also $\tilde{\mathbf{g}} = \phi_* \mathbf{g} = \mathbf{h}$, is called an *isometry*.

As illustrated in Figure 3.1, an active diffeomorphism is intended to transform the objects representing the physical world, but keep the reference system unchanged. This requires the target manifold to be the same as the original one, $\mathcal{N} = \mathcal{M}$, and the system of local coordinates after the mapping to be the same

as before, $\tilde{\psi} = \psi$. The relation between old and new local coordinate values is defined above as $\tilde{x} = \theta(x)$ where $\theta = \tilde{\psi} \circ \phi \circ \psi^{-1}$. When $\tilde{\psi} = \psi$, this becomes

$$\theta = \psi \circ \phi \circ \psi^{-1} \tag{3.20}$$

which is a defining property of any active diffeomorphism. Note that, unlike the passive case in Section 3.1 which only changed the local coordinates while leaving the underlying manifold objects unchanged, active diffeomorphisms change the underlying manifold objects in $\mathcal{M}$ to new underlying manifold objects in the same manifold $\mathcal{M}$.

3.3 Active Diffeomorphism with Fixed Metric

In the pre-general-relativistic context of standard differential geometry, metric $\mathbf{g}$ is a fixed part of the definition of a Riemannian manifold, denoted $(\mathcal{M}, \mathbf{g})$, and the metric $\mathbf{h}$ is a fixed part of the definition of the target manifold $(\mathcal{N}, \mathbf{h})$. Since active diffeomorphisms are automorphisms with $\mathcal{N} = \mathcal{M}$, and since a metric is fixed to its manifold, it must also be true that $\mathbf{h} = \mathbf{g}$. Thus the mapping is

$$\phi : (\mathcal{M}, \mathbf{g}) \to (\mathcal{M}, \mathbf{g}) \tag{3.21}$$

This means that only *isometric* active diffeomorphisms are allowable in this pre-general-relativistic context, those with $\tilde{\mathbf{g}} = \phi_*\mathbf{g} = \mathbf{g}$.[19]

3.4 Active Diffeomorphism in General Relativity

In general relativity the metric is not a fixed, prescribed property of a Riemannian manifold. It is the solution of a differential equation, unknown until the equation is solved. Thus, as developed by Wald (1984) and Carroll (2016), in general relativity the metric tensor can be transformed arbitrarily in active diffeomorphisms, just as one would transform any other second rank, covariant tensor field. The active diffeomorphism can be represented in standard notation as

$$\phi : (\mathcal{M}, \mathbf{g}) \to (\mathcal{M}, \phi_*\mathbf{g}) \tag{3.22}$$

If $\phi_*\mathbf{g} = \mathbf{g}$ we are of course back to the isometric transformations of eqn (3.21).[20] Although $\mathbf{g}$ may transform in the active diffeomorphisms of general relativity, they remain automorphisms from manifold $\mathcal{M}$ to itself without change of local coordinate system; thus eqn (3.20) must remain true for them.

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[19]Due to the condition $\tilde{\psi} = \psi$ in eqn (3.20), the local coordinate expression of this isometry is $\tilde{g}_{ij}(\tilde{x}) = g_{ij}(\tilde{x})$.

[20]Appendix 5 of Carroll (2016) refers to the mappings in eqn (3.22) simply as "diffeomorphisms" or, occasionally, "active diffeomorphisms." Thus Carroll's term "diffeomorphism" is a synonym of "active diffeomorphism." Also, Wald and Carroll refer to passive diffeomorphisms as gauge transformations.

3.5 Generation of Active Diffeomorphisms

Some of the machinery of Lie Group theory can be borrowed to generate active diffeomorphisms from given tangent vector fields. A useful class of active diffeomorphisms can be constructed by considering the mapping ϕ_τ along a given tangent vector field $V(x)$.[21] Given a chosen starting point $p \in \mathcal{M}$, a smooth mapping $\eta : (0, \tau_1) \to \mathcal{M}$ defines a curve in $\mathcal{M}$, with $p(\tau) = \eta(\tau)$ and $p(0) = \eta(0) = p$. In terms of local coordinates this is $x(\tau) = \psi(p(\tau))$. Differentiating this curve with respect to τ gives what is sometimes called a "velocity" tangent vector $W(x(\tau))$ along the curve. Its components are $\dot{x}^i(\tau) = dx^i(\tau)/d\tau$. Given a general tangent vector field $V(x)$, a curve whose velocity matches that tangent vector for every $\tau \in (0, \tau_1)$ is defined by the set of differential equations

$$\dot{x}^i(\tau) = V^i(x(\tau)) \qquad \text{where} \qquad i = 1, \dots, m \tag{3.23}$$

whose solution $x(\tau)$ can be described as an integral curve or "field line" of $V(x)$ passing through $x(0)$. The corresponding field line in the manifold is then $p(\tau) = \psi^{-1}(x(\tau))$.

Since the tangent vector field is assumed to be defined at all points of $\mathcal{M}$, we can consider the family of all such field-line curves beginning at every point $p \in M$. Consider an active diffeomorphic mapping $\phi_\tau : \mathcal{M} \to \mathcal{M}$ which simultaneously carries each $p = p(0)$ in $\mathcal{M}$ into a $\tilde{p} = p(\tau)$ along the particular field line starting at p. When $\tau = 0$, this mapping is the identity mapping $\phi_0 = I$. When $\tau > 0$, mapping ϕ_τ will move each point $p = p(0)$ of $\mathcal{M}$ along the appropriate field line to a new point $\tilde{p} = p(\tau) = \phi_\tau(p)$. Expressing the same mapping in local coordinates, each point $x = x(0) = \psi(p)$ is moved by active diffeomorphism $\theta_\tau = \psi \circ \phi_\tau \circ \psi^{-1}$ into a new point $\tilde{x} = x(\tau) = \theta_\tau(x)$. It is important that the mapping ϕ_τ is smoothly connected to the identity at $\tau = 0$. This ensures that the generated active diffeomorphisms based on ϕ_τ do not involve a change of coordinate scheme that would violate eqn (3.20).

If $V(x)$ is a Killing Vector Field, then, by definition the active diffeomorphism ϕ_τ is isometric. Generation of more general active diffeomorphisms with $\tilde{\mathbf{g}} = \phi_*\mathbf{g} \neq \mathbf{g}$ requires that $V(x)$ not be a Killing Vector Field.

3.6 Examples of the Generation of Active Diffeomorphisms

Consider Cartesian three space with coordinates[22] (x, y, z) and metric defined by the matrix $\mathsf{g}(x) = \mathrm{diag}(1, 1, 1)$.

Choose a Killing Vector Field with components $V(x) = (-y, x, 0)$. Then eqn (3.23) becomes

$$\frac{dx(\tau)}{d\tau} = -y \qquad \frac{dy(\tau)}{d\tau} = x \qquad \frac{dz(\tau)}{d\tau} = 0 \tag{3.24}$$

with solution

$$\tilde{x} = x(\tau) = A\cos\tau - B\sin\tau \qquad \tilde{y} = y(\tau) = A\sin\tau + B\cos\tau \qquad \tilde{z} = C \tag{3.25}$$

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[21]Section 39 of Arnold (1978), pages 68-70 and Chapter 9 of Lee (2013), and pages 27-32 and 250-251 of O'Neill (1983).

[22]The coordinates $x = (x^1, x^2, x^3)$ are written here as (x, y, z) for readability.

The initial condition $(x(0), y(0), z(0)) = (x, y, z)$ then gives the active diffeo-morphism for epoch τ as

$$\tilde{x} = x \cos \tau - y \sin \tau \qquad \tilde{y} = x \sin \tau + y \cos \tau \qquad \tilde{z} = z \qquad (3.26)$$

which is the same as eqn (3.3) with epoch τ identified with rotation angle α. Since this transformation is orthogonal, the transformed metric remains $\tilde{\mathbf{g}}(\tilde{x}) = \mathrm{diag}(1, 1, 1)$. Thus $\tilde{\mathbf{g}} = \phi_*\mathbf{g} = \mathbf{g}$ and the active diffeomorphism is isometric.

Now choose a non-Killing Vector Field with components $V(x) = (y, 0, 0)$. Then eqn (3.23) becomes

$$\frac{dx(\tau)}{d\tau} = y \qquad \frac{dy(\tau)}{d\tau} = 0 \qquad \frac{dz(\tau)}{d\tau} = 0 \qquad (3.27)$$

with solution

$$\tilde{x} = x(\tau) = a + b\tau \qquad \tilde{y} = y(\tau) = b \qquad \tilde{z} = c \qquad (3.28)$$

The initial condition $(x(0), y(0), z(0)) = (x, y, z)$ then gives the active diffeomor-phism for epoch τ as

$$\tilde{x} = x + y\tau \qquad \tilde{y} = y \qquad \tilde{z} = z \qquad (3.29)$$

The transformed metric obtained from the inverse of eqn (3.19) is

$$\tilde{\mathbf{g}}(\tilde{x}) = \begin{pmatrix} 1 & -\tau & 0 \\ -\tau & (\tau^2 + 1) & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad (3.30)$$

and the active diffeomorphism is not isometric.

Note that both of these active diffeomorphisms reduce smoothly to the iden-tity when $\tau = 0$, consistent with eqn (3.20) and the condition that active diffeo-morphisms do not change the system of local coordinates but only the manifold point being represented.

3.7 Essential Difference Between Passive and Active Diffeo-morphisms

Passive diffeomorphisms change the system of local coordinates but do not change the manifold objects being represented in those coordinates. Thus the same point $p \in \mathcal{M}$ in the manifold is represented by $x = \psi(p)$ and $x' = \psi'(p)$. Also $g_{\mu\nu}(x)$ and $g'_{\mu\nu}(x')$ both represent the same underlying metric tensor field $\mathbf{g}(p)$ defined on the manifold $\mathcal{M}$.

Active diffeomorphisms are the opposite of passive ones. In them the un-derlying manifold point $p \in \mathcal{M}$ is mapped to a different point $\tilde{p} = \phi_\tau(p) \in \mathcal{M}$ with both old and new points being represented in the same system of local coordinates. Thus $x = \psi(p)$ and $\tilde{x} = \psi(\tilde{p})$ are different, not because of a change of coordinate system, but because the manifold point being represented has been mapped from p to $\tilde{p}$. The underlying manifold objects are also changed: $\tilde{f}(p) \neq f(p)$, $\tilde{\mathbf{V}}(p) \neq \mathbf{V}(p)$, $\tilde{\mathbf{g}}(p) \neq \mathbf{g}(p)$, and so on for other tensors. Manifold ob-jects are often used to model the physical world. The change of these manifold

objects by an active diffeomorphism thus changes the model. The thrust of the hole argument in Section 4 is to prove that two different models, one derived from the other by an active diffeomorphism, can both be solutions of the same Einstein field equation.

4 Einstein's Hole Argument in General Relativity

The Einstein field equation may be written as

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \kappa T_{\mu\nu} = 0 \tag{4.1}$$

where $R_{\mu\nu}$ is the Ricci tensor, R is the curvature invariant, κ is a universal constant related to the Newton gravitational constant, and $T_{\mu\nu}$ is the energy-momentum tensor source. This equation may be written in a form that presents its dependency on the metric tensor and its derivatives explicitly. It is

$$\mathcal{R}_{\mu\nu}(g(x), x) - \frac{1}{2}g_{\mu\nu}(x)\left(g^{\alpha\beta}(x)\mathcal{R}_{\alpha\beta}(g(x), x)\right) + \kappa T_{\mu\nu}(x) = 0 \tag{4.2}$$

where the functions $\mathcal{R}_{\mu\nu}$ are defined by[23]

$$\mathcal{R}_{\mu\nu}(g(x), x) = \frac{\partial}{\partial x^\nu}\left\{{\alpha \atop \mu\alpha}\right\} - \frac{\partial}{\partial x^\alpha}\left\{{\alpha \atop \mu\nu}\right\} + \left\{{\beta \atop \mu\alpha}\right\}\left\{{\alpha \atop \beta\nu}\right\} - \left\{{\beta \atop \mu\nu}\right\}\left\{{\alpha \atop \beta\alpha}\right\} \tag{4.3}$$

$$\text{where} \quad \left\{{\mu \atop \nu\alpha}\right\} = \frac{1}{2}g^{\mu\delta}(x)\left(\frac{\partial g_{\delta\nu}(x)}{\partial x^\alpha} + \frac{\partial g_{\delta\alpha}(x)}{\partial x^\nu} - \frac{\partial g_{\nu\alpha}(x)}{\partial x^\delta}\right) \tag{4.4}$$

Now perform a general non-Killing active diffeomorphism $\theta_\tau = \psi \circ \phi_\tau \circ \psi^{-1}$ from local coordinates x to local coordinates $\tilde{x} = \theta_\tau(x)$. After this active diffeomorphism the field equation becomes

$$\mathcal{R}_{\mu\nu}(\tilde{g}(\tilde{x}), \tilde{x}) - \frac{1}{2}\tilde{g}_{\mu\nu}(\tilde{x})\left(\tilde{g}^{\alpha\beta}(\tilde{x})\mathcal{R}_{\alpha\beta}(\tilde{g}(\tilde{x}), \tilde{x})\right) + \kappa\tilde{T}_{\mu\nu}(\tilde{x}) = 0 \tag{4.5}$$

Comparing eqn (4.5) after the active diffeomorphism to eqn (4.2) before it, note that there is no tilde on the function $\mathcal{R}$. Due to eqn (3.22) and the general covariance of the Ricci tensor, $\mathcal{R}_{\mu\nu}(\tilde{g}(\tilde{x}), \tilde{x})$ is exactly the same function of $(\tilde{g}(\tilde{x}), \tilde{x})$ as $\mathcal{R}_{\mu\nu}(g(x), x)$ is of $(g(x), x)$. But, since we have not yet applied Einstein's restricted definition of the energy-momentum source, $\tilde{T}_{\mu\nu}$ is generally a different function from $T_{\mu\nu}$.

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[23]See Synge and Schild (1978) equations 2.241, 2.242, and 3.203. The Einstein summation convention is used. A term containing a repeated Greek index is summed over that index, from 0 to 3.
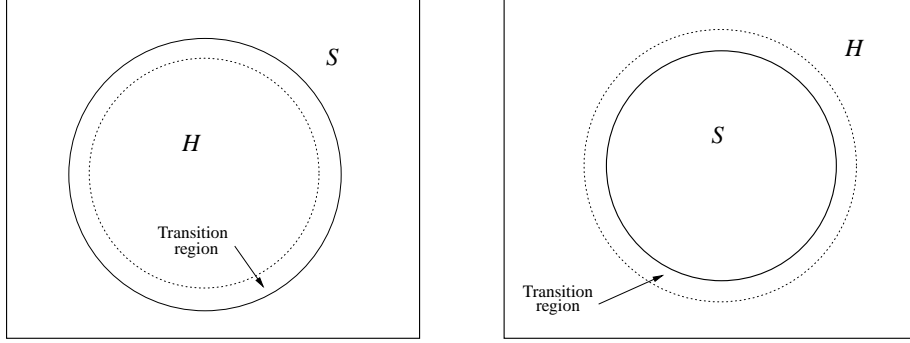
Figure 4.1: A schematic diagram of the hole H and source S regions of the hole argument. On the left the hole H is surrounded by source S as envisioned by Einstein. On the right, the source S is surrounded by the hole H as used in Section 7. The transition regions inside H are necessary to preserve the differentiability of the active diffeomorphism.

Now apply Einstein's restrictions. First, assume only experimental situations in which there is a hole region H with $T_{\mu\nu}(x) \equiv 0$ for $x \in H$. Then consider an active diffeomorphism that is the identity ($V(x) \equiv 0$ and hence $\theta_\tau(x) = I$) in the region S that is the complement of H, but not the identity in H itself.[24] It follows that the energy-momentum tensor is untransformed both in H (because a zero tensor transforms to the zero tensor regardless of the transformation) and S (because the active diffeomorphism is the identity in S). Thus $\tilde{T}_{\mu\nu}(\tilde{x}) = T_{\mu\nu}(\tilde{x})$ throughout the manifold, and eqn (4.5) becomes

$$\mathcal{R}_{\mu\nu}(\tilde{g}(\tilde{x}), \tilde{x}) - \frac{1}{2}\tilde{g}_{\mu\nu}(\tilde{x})\left(\tilde{g}^{\alpha\beta}(\tilde{x})\mathcal{R}_{\alpha\beta}(\tilde{g}(\tilde{x}), \tilde{x})\right) + \kappa T_{\mu\nu}(\tilde{x}) = 0 \qquad (4.6)$$

with no tilde on the T. However, the function $\tilde{g}_{\mu\nu}$ that solves eqn (4.6) is not the same function as the $g_{\mu\nu}$ that solves eqn (4.2). Inside the hole region $\tilde{g}_{\mu\nu}(\tilde{x}) \neq g_{\mu\nu}(\tilde{x})$.

If eqn (4.6) is satisfied, it follows that the differential equation

$$\mathcal{R}_{\mu\nu}(\tilde{g}(x), x) - \frac{1}{2}\tilde{g}_{\mu\nu}(x)\left(\tilde{g}^{\alpha\beta}(x)\mathcal{R}_{\alpha\beta}(\tilde{g}(x), x)\right) + \kappa T_{\mu\nu}(x) = 0 \qquad (4.7)$$

must also be satisfied. Comparison of eqn (4.7) and eqn (4.2) demonstrates that $\tilde{g}_{\mu\nu}(x)$ and $g_{\mu\nu}(x)$ are both solutions to the same Einstein field equation. Thus, in this experimental situation, there are two or more solutions to the Einstein field equation with the same energy-momentum tensor source, as Einstein asserted. The physical meaning of these multiple solutions, and the possibility of the rejection of some of them as spurious, is the subject of Section 5 below.

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[24]See Figure 4.1. The chosen active diffeomorphism must have a small transition region just inside H, transitioning smoothly from identity in S to non-identity inside H, in order to satisfy the basic condition that active diffeomorphisms must be smooth functions.

Note that eqn (4.7) differs from eqn (4.6) only in the replacement $\tilde{x} \to x$ throughout. The argument leading from eqn (4.6) to eqn (4.7) is as follows: In linear algebra, one often uses dummy indices whose replacement by other letters does not change a sum, provided that the two sets of indices are summed over the same range. Thus the equality $\sum_{i=1}^{5} K_i = 6$ is true if and only if $\sum_{j=1}^{5} K_j = 6$ is true. Dummy indices have an analog in differential equations. The equality $df(t)/dt = -\lambda f(t)$ is true if and only if $df(u)/du = -\lambda f(u)$ is true, provided only that the dummy variables t and u are of the same character and range, here real numbers in $(-\infty, \infty)$. Now make the same sort of substitution in eqn (4.6), with $\tilde{x}$ in place of t and x in place of u, with $\tilde{g}_{\mu\nu}$ playing the role of f. The condition that $\tilde{x}$ and x are variables of the same character and range is ensured by the condition $\tilde{\psi} = \psi$ and eqn (3.20). Also, the fact that active diffeomorphisms constructed as in Section 3.5 are smoothly connected to the identity when $\tau \to 0$ rules out transformations, such as from Cartesian to spherical polar, that would make the ranges of $\tilde{x}$ and x different. With the substitution $\tilde{x} \to x$, the equality in eqn (4.6) is true if and only if the equality in eqn (4.7) is true.

Note the crucial importance of Einstein's restriction that the energy-momentum source must vanish in the hole. Without that restriction, the T in both eqn (4.6) and eqn (4.7) would be replaced by $\tilde{T}$. The $\tilde{g}$ would still be a different metric solution, but it would be the solution to a *different* differential equation, one with an actively transformed source $\tilde{T}$ that models a *different* experimental situation, and not a second solution to the original differential equation with the original source T. Without the Einstein condition on T, the above proof of multiple solutions fails.

5 Physical Meaning[25] in Einstein's Multiple Metrics

Einstein's final form of his field equation is generally covariant. It therefore suffers from the multiplicity of solutions derived in Section 4. His resolution was to assert that all the metric solutions are physically equivalent, and to deny that local coordinates represent anything real.[26]

In one reading, Einstein's denial of the reality of local coordinates only repeats a fact of pre-general-relativistic differential geometry. The local coordinates $x = \psi(p)$ with $p \in \mathcal{M}$ defined on a bare manifold $\mathcal{M}$ by means of homeomorphism ψ do not initially have any definite relation to any physical or geometrical quantity. The coordinates x are just m-tuples of real numbers. In pre-general-relativistic cases, these numbers acquire geometric or physical meaning only when the move is made to Riemannian geometry by adding a (fixed) metric to the manifold.[27] The situation is even more extreme in general

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[25]As noted above, by "physical meaning" of local coordinates I mean a defined relation between them and some physical quantity like length or relativistic interval.

[26]See "How Einstein Discovered General Relativity: A Historical Tale With Some Contemporary Morals" pp. 293-299 of Stachel (2002)

[27]Of course the natural Euclidean metric of m-tuples of real numbers is always available. But it does not have to be applied. For example, in Hamiltonian mechanics the Euclidean metric is not

relativity, in which a definite metric is not even available to be applied to $\mathcal{M}$ until *after* the field equation is solved. In general relativity, a solution to the Einstein field equation is to be obtained using local coordinates of unknown physical meaning. Each metric tensor solution then has a privileged role; each of them determines the physical meaning of the coordinates x in terms of which it is written.

Since local coordinates obtain their physical meaning only from a metric solution to the field equation, it follows that different metric solutions to the field equations may give different physical meanings to the same set of local coordinates. The fact that, at a given manifold point p, the local coordinates $x = (x^0, x^1, x^2, x^3)$ in $g_{\mu\nu}(x)$ are the same quadruple of real numbers as the x in $\tilde{g}_{\mu\nu}(x)$ does not mean that the local coordinates x have the same physical meaning in both solutions. Each metric solution brings its own assignment of physical or geometrical meaning to the local coordinates used to write it.

On this reading, Einstein's statement should be modified to say not that local coordinates have no meaning,[28] but rather that local coordinates have no *independent* meaning, independent of the metric solution. Each of the multiple metric solutions carries its own physical interpretation of its own local coordinates. I propose three resolutions to this problem of undetermined local coordinate meaning, each of which denies the necessity of multiple solutions to the field equation.

5.1 Resolution A: Active Diffeomorphisms Must be Isometric

Resolution A suggests that a strict definition of the term "active diffeomorphisms" requires them to be isometric, and thus prevents their use in the hole argument. The condition $\tilde{\psi} = \psi$ leading to eqn (3.20) was to guarantee that the mapping $\psi(p)$ from manifold points p to local coordinates is the same before and after the active diffeomorphism. This is the defining condition that an active diffeomorphism modifies the physical world but must not modify the system of coordinates used to observe it. But when the non-isometric case $\tilde{\mathbf{g}} \neq \mathbf{g}$ is allowed, the transformed metric $\tilde{\mathbf{g}}(\tilde{x})$ gives transformed coordinate $\tilde{x}$ a physical meaning different from the one that the original metric $\mathbf{g}(x)$ gave to original coordinate x. This difference of meaning modifies the system of local coordinates in an essential way; it therefore violates the defining condition of active diffeomorphisms and must be rejected. But when only isometric active diffeomorphisms are allowed, there is no hole argument. By definition, an isometric active diffeomorphism simply replicates the same metric tensor and does not provide a new one; multiple solutions are not generated.

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used, with a symplectic structure function applied instead.

[28]Note that solutions to the Einstein field equation such as the Schwarzschild or Robertson-Walker metrics do indeed assign a physical meaning to their coordinates.

5.2 Resolution B: Selection by Symmetry

Resolution B ignores the strict definition demanded by Resolution A and allows non-isometric active diffeomorphisms of the sort outlined in Section 3 and Carroll (2016). Each of the resulting multiple solutions to eqn (4.2) is then an equally valid candidate solution, but each gives a different physical meaning to the local coordinates in terms of which it is written. As with many differential equations, one must then reject some solutions as spurious. Any solution that gives a physical meaning to its local coordinates that violates the symmetries demanded by the experimental situation being modeled can be rejected as spurious. Thus solving eqn (4.2) is only the first step in a solution procedure for the Einstein field equation. There are multiple solutions, but also a method to select the correct one from among them and to reject the others as spurious.

5.3 Resolution C: Use of a Template

Resolution C is similar to Resolution B above, but rather than actually choosing one solution with the desired symmetry from a multiplicity of candidate solutions, one simply enforces symmetry from the start by specifying a template that forces a single solution exhibiting that symmetry. For example, in Section 6 for the Schwarzschild metric one solves the Einstein equation in two steps, the first of which is to choose a template metric which forces spherical symmetry and almost completely defines the physical meanings of its coordinates. The second step is to substitute this template into the field equation to determine its remaining parameters. In effect, the metric is largely determined by the first step; the Einstein field equation is used as a kind of auxiliary condition to determine certain residual parameters and ensure consistency with Newtonian gravity. Solutions other than the one derived from the template then violate the template and its symmetry and can be rejected as spurious.

Although Resolutions A, B, and C differ, they agree that, despite the mathematical proof in Section 4, the existence of a unique solution to the Einstein field equation cannot be ruled out.

6 The Schwarzschild Example

A good test case to illustrate the hole argument with active diffeomorphisms that modify the metric tensor is the Schwarzschild solution to the Einstein field equation in the empty space surrounding a spherically symmetric source region.

The first step to the Schwarzschild solution is to construct a template metric that defines the geometric properties of some of the local coordinates and enforces spherical symmetry.[29] A standard template denotes the variable set as $(x^0, x^1, x^2, x^3) = (t, r, \theta, \phi)$ and sets the template $g_{\mu\nu}(x)$ equal to the diagonal

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[29]See Chapter 8 of Weinberg (1972), Chapter 11 of Rindler (2006), Chapter 14 of d'Inverno (1992).

matrix

$$g(x) = \text{diag}\left(-c^2\beta(r), \alpha(r), r^2, r^2\sin^2\theta\right) \tag{6.1}$$

This choice of template enforces the spherical symmetry of the problem, identifies θ and ϕ as the standard angles of spatial spherical polar coordinates, and makes the area of the surface $t = \text{const.}$, $r = \text{const.}$ equal to $4\pi r^2$. This template is substituted into eqn (4.2); straightforward algebra then determines the functions α and β and arrives at

$$g(x) = \text{diag}\left\{-\left(1 - \frac{2m}{r}\right)c^2, \left(1 - \frac{2m}{r}\right)^{-1}, r^2, r^2\sin^2\theta\right\} \tag{6.2}$$

where $m = GM/c^2$, with Newton's gravitational constant G, the total mass of the source M, and the speed of light c.[30] The Schwarzschild solution in eqn (6.2) is *uniquely determined* given the template that sets its desired symmetry.

The Robertson-Walker metric is similarly derived from a template enforcing its symmetries[31]

$$g(x) = \text{diag}\left(-c^2, (\alpha(t))^2/(1 - kr^2), (r\alpha(t))^2, (r\alpha(t))^2\sin^2\theta\right) \tag{6.3}$$

where k is -1, 0, or $+1$ and scale factor $\alpha(t)$ can be derived from the Einstein field equation together with assumptions about the density and nature of matter in a cosmological model.[32]

7   The Hole Argument with the Schwarzschild Solution

Now apply the hole argument to the Schwarzschild solution. Referring to Figure 4.1 and the description of the hole argument in Section 4, region S can be taken as all points with $r \leq r_1$ where r_1 is a radius beyond all energy-momentum tensor sources and also beyond the Schwarzschild radius $r = 2m$. Region H is then all points with $r > r_1$. The transition region inside H is all points with $r_1 < r < r_2$ where r_2 is some arbitrarily chosen boundary. In this example, the "hole" region H is in fact exterior to the source region S, but this choice makes no difference to the hole argument. All that is required is that $H \cap S = \emptyset$ and $H \cup S = \mathcal{M}$.

To apply the hole argument, first define a smoothing function to enforce the

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[30]If the unknown functions α and β in eqn (6.1) are allowed to be functions of both *r and t*, the same metric eqn (6.2) is obtained, a result known as Birkhoff's theorem. See Chapter XV of Birkhoff and Langer (1923), and Chapter 14 of d'Inverno (1992).

[31]Chapter 13 of Weinberg (1972) derives this template as well as eqn (6.1) from a requirement of maximal subspace symmetry, with no prior reference to the Einstein field equation.

[32]See d'Inverno (1992), Section 22.9.

differentiability of the active diffeomorphism in the transition region. It is[33]

$$\xi(s) = \begin{cases} \exp(-1/s) & \text{for} \quad s > 0 \\ \\ 0 & \text{for} \quad s \leq 0 \end{cases} \tag{7.1}$$

Then choose an arbitrary (but non-Killing) tangent vector field $X(x)$ and define a tangent vector field $V(x)$ by

$$V(x) \equiv 0 \qquad V(x) = \left(\frac{\xi(r - r_1)}{\xi(r - r_1) + \xi(r_2 - r)} \right) X(x) \qquad V(x) = X(x) \tag{7.2}$$

in region S, the transition region, and the remainder of region H, respectively.

As described in Section 3.5, for any fixed τ value an active diffeomorphism ϕ_τ can be defined by following the field lines of tangent vector $V(x)$. It changes the metric of eqn (6.2) to a new metric in region H, but without changing the metric or the source in region S where $V(x) \equiv 0$ and hence $\phi_\tau = I$, the identity transformation.

Applying the hole argument with this active diffeomorphism, the unchanged mass source in region S now produces a family of different metrics $\tilde{\mathbf{g}} = \phi_{\tau*}\mathbf{g}$ that solve eqn (4.2) in region H. It then follows that a given source in region S of the Schwarzschild problem produces many different solutions in region H, one for each τ value and choice of tangent vector field $X(x)$.

For example, choose the $X(x)$ to have local coordinates $(0, a\phi, 0, 0)$ where a is some fixed parameter having units of length. Use the inverse of eqn (3.19) with the active diffeomorphism given by the procedure in Section 3.5 to write the transformed metric tensor. In region H beyond the transition region, it is

$$\tilde{g}(x) = \begin{pmatrix} -c^2\lambda & 0 & 0 & 0 \\ 0 & \lambda^{-1} & 0 & -a\tau\lambda^{-1} \\ 0 & 0 & (r - a\tau\phi)^2 & 0 \\ 0 & -a\tau\lambda^{-1} & 0 & \chi \end{pmatrix} \tag{7.3}$$

where

$$\lambda = \frac{r - a\tau\phi - 2m}{r - a\tau\phi} \qquad \text{and} \qquad \chi = a^2\tau^2\lambda^{-1} + (r - a\tau\phi)^2 \sin^2\theta \tag{7.4}$$

Because of its generation by the hole procedure, eqn (7.3) is certainly another solution to the Einstein field equation with the same source field T. But it may be rejected by symmetry considerations. The Schwarzschild solution eqn (6.2) enforces the desired spherical symmetry resulting from the assumed spherically symmetric mass distribution. The metric solution eqn (7.3) lacks that spherical symmetry. In fact, due to the uniqueness of the Schwarzschild solution given the template metric, there is no possible alternate solution with the same template eqn (6.1) but different parameters α and β. The Schwarzschild

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[33]See pages 40-42 of Lee (2013).

metric stands as a counterexample to the proposition that the Einstein field equation must of necessity always have multiple solutions.

The Robertson-Walker metric is similarly derived by starting with a template, eqn (6.3), enforcing spherical symmetry. A non-isometric active diffeomorphism applied to it will result in a metric that violates that template, just as in the Schwarzschild case.

8 Conclusion

The proof in Section 4 demonstrates mathematically that the Einstein field equation has multiple metric solutions. But since no metric is defined until after the field equation is solved, that proof is of necessity just a numerical exercise written using local coordinates that are quadruples of real numbers with no definite physical meaning, i.e., no assigned relation to relativistic interval. After the field equation is solved, each of the multiple metric solutions then assigns its own physical meaning to the local coordinates in terms of which it is written. As noted in Resolutions B and C of Section 5, these various physical meanings of the local coordinates, as read from the various metric solutions, may then be used to reject as spurious those solutions whose local coordinates have a meaning inconsistent with the symmetries of the experimental situation being modeled. This rejection of spurious solutions opens the possibility that in some cases, such as the Schwarzschild metric, only one solution may survive. Thus the hole argument cannot prove the assertion that the Einstein field equation must have multiple solutions.

A considerable intellectual superstructure has been built on the foundation of the hole argument, beginning with Einstein himself who asserted that because of it the local coordinates used to write his field equation can have no physical meaning.[34] Later authors[35] have expanded this intuition into a general argument against what is sometimes called manifold substantivalism, roughly defined as a realist interpretation of the manifold of differential geometry. The failure of the hole proof noted above removes Einstein's contribution to this argument.

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[34]See Section 5.

[35]Earman and Norton (1987), Chapter V of Stachel (2002) and others quoted therein.

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