Magnetic Charge and Dyality Invariance

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Abstract

This paper is a critical study of non-standard Maxwellian electrodynamics. It explores two important topics: the inclusion of both magnetic and electric charge to produce what it calls *Extended Electrodynamics,* and the existence of a symmetry called *Dyality Invariance* that exchanges electric and magnetic quantities.

First, the paper summarizes *Extended Electrodynamics,* including potentials, gauge transformations, and a new proof of the extended electrodynamic Poynting theorem.

A formal Lagrangian derivation of the extended Maxwell equations is also given, but its value in fundamental studies is questioned.

The paper then defines *Dyality Invariance* (form invariance under the so-called *Dyality Transformation* that exchanges electric and magnetic quantities) and shows it to be a valid symmetry if and only if electrodynamics is given the *extended* form.

The paper suggests that the complete Maxwellian electrodynamics is extended electrodynamics with its dyality invariance. But dyality can be interpreted either *actively* or *passively.* Since magnetic charge has not been observed experimentally, the active interpretation is ruled out. But a passive interpretation can be used to avoid writing magnetic source and potential terms explicitly.

The paper also refutes the idea that dyality invariance would permit a magnetic charge to be transformed away even if one existed. If nonzero magnetic charge exists, then experimental evidence for its existence cannot be hidden by a dyality transformation.

1 Preface

This paper explores two important topics: the inclusion of both magnetic and electric charge to produce what is called *Extended Electrodynamics,* and the existence of a symmetry called *Dyality Invariance* that exchanges electric and magnetic quantities.[1]

Sections 2 through 5 summarize *Extended Electrodynamic*s, including potentials and gauge transformations. A new proof of the extended electrodynamic Poynting theorem is given in Section 3 and Appendix A in Section 11.

Section 6 gives a formal Lagrangian derivation of extended electrodynamics, but suggests that Lagrangian methods are of limited value.

In Section 7, *Dyality Invariance* (form invariance under an exchange of electric and magnetic quantities called a *Dyality Transformation*) is defined and shown to be a symmetry *only* of extended electrodynamics. Standard electrodynamics with only electric charge is not invariant under dyality transformations.

Sections 7 and 8 demonstrate the dyality invariance of extended electrodynamics, including the extended Maxwell equations themselves, the energy-momentum tensor, and expressions involving the four-vector potentials.

Section 9 introduces the distinction between *active* and *passive* interpretations of dyality transformations. Active interpretation of dyality invariance would imply the experimental existence of magnetic charge, and is therefore currently ruled out. However, since magnetic charge has not yet been found experimentally, a passive interpretation of dyality invariance allows us to consider the standard electrodynamics to be extended electrodynamics with the magnetic sources and potentials transformed passively out of all equations.

Section 9 also offers a refutation of the idea that dyality invariance would permit an *experimentally existing* magnetic charge to be transformed away. If nonzero magnetic charge exists, experimental evidence for its existence cannot be hidden by any dyality transformation.

An afterword in Section 10 suggests directions for future research. The paper has two Appendices.

2 Notation and Definitions

This section introduces some definitions to be used in the paper.

[1] The neologism "dyality" is suggested by [6] to prevent confusion with "duality", the relation, for example, in eq.(2.10).

An event x^μ is denoted by

$$x^0 = ct \qquad x^\mu = (x^0, x^1, x^2, x^3)$$
$$\partial_\mu = \left(\frac{\partial}{\partial x^0}, \frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}, \frac{\partial}{\partial x^3}\right) \qquad \Box^2 = \partial_\mu \partial^\mu \tag{2.1}$$

We denote four-vectors as $\mathsf{K} = K^0 \mathbf{e}_0 + \mathbf{K}$ where $\mathbf{e}_0$ is the time unit vector and the three-vector part is understood to be $\mathbf{K} = K^1 \mathbf{e}_1 + K^2 \mathbf{e}_2 + K^3 \mathbf{e}_3$. In the Einstein summation convention, Greek indices range from 0 to 3, Roman indices from 1 to 3. The Minkowski metric tensor used to raise or lower indices is $\eta_{\mu\nu} = \eta^{\mu\nu} = \mathrm{diag}(-1, +1, +1, +1)$. Three-vectors are written with bold type $\mathbf{K}$, and their magnitudes as K. The paper uses Heaviside-Lorentz electromagnetic units, and considers only electrodynamics in a vacuum except for explicit source charge densities.

The completely antisymmetric, Minkowski-space, Levi-Civita tensor $\varepsilon^{\alpha\beta\mu\nu}$ obeys the identities

$$\varepsilon^{0123} = +1 \qquad \varepsilon_{0123} = -1 \tag{2.2}$$
$$\varepsilon^{\alpha\beta\mu\nu}\varepsilon_{\alpha\beta\gamma\delta} = -2\left(\delta^\mu_\gamma \delta^\nu_\delta - \delta^\mu_\delta \delta^\nu_\gamma\right) \tag{2.3}$$
$$\varepsilon^{\alpha\beta\mu\nu}\varepsilon_{\alpha\theta\gamma\delta} = -\left(\delta^\beta_\theta \delta^\mu_\gamma \delta^\nu_\delta + \delta^\beta_\gamma \delta^\mu_\delta \delta^\nu_\theta + \delta^\beta_\delta \delta^\mu_\theta \delta^\nu_\gamma\right) + \left(\delta^\beta_\theta \delta^\mu_\delta \delta^\nu_\gamma + \delta^\beta_\delta \delta^\mu_\gamma \delta^\nu_\theta + \delta^\beta_\gamma \delta^\mu_\theta \delta^\nu_\delta\right) \tag{2.4}$$

where $\delta^\alpha_\beta = \eta^{\alpha\mu}\eta_{\beta\mu}$ is the Kroeneker-delta tensor, which is $+1$ when $\alpha = \beta$ and zero otherwise.

If $X^{\alpha\beta}$ is an antisymmetric second rank tensor, then its *dual* $\tilde{X}^{\alpha\beta}$ is another antisymmetric second rank tensor, defined as

$$\tilde{X}^{\alpha\beta} = \frac{1}{2}\varepsilon^{\alpha\beta\mu\nu} X_{\mu\nu} \tag{2.5}$$

It follows from eq.(2.3) that

$$\tilde{\tilde{X}}^{\alpha\beta} = \frac{1}{2}\varepsilon^{\alpha\beta\mu\nu}\tilde{X}_{\mu\nu} = \frac{1}{4}\varepsilon^{\alpha\beta\mu\nu}\varepsilon_{\mu\nu\theta\delta}X^{\theta\delta} = -\frac{1}{2}\left(\delta^\alpha_\theta \delta^\beta_\delta - \delta^\alpha_\delta \delta^\beta_\theta\right) X^{\theta\delta} = -X^{\alpha\beta} \tag{2.6}$$

Other useful identities follow from eqs.(2.4 and 2.5). If $X^{\alpha\beta}$ and $Y^{\alpha\beta}$ are antisymmetric tensors, then

$$\tilde{X}^{\alpha\mu}\tilde{Y}_{\alpha\nu} = Y^{\alpha\mu}X_{\alpha\nu} - \frac{1}{2}\delta^\mu_\nu X^{\alpha\beta}Y_{\alpha\beta} \tag{2.7}$$
$$\tilde{X}^{\alpha\beta}\tilde{Y}_{\alpha\beta} = -X^{\alpha\beta}Y_{\alpha\beta} \tag{2.8}$$

The Maxwell field four-tensor $F^{\alpha\beta}$ in terms of the electric field $\mathbf{E}$ and the magnetic field $\mathbf{B}$ is

$$F^{\mu\nu} = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & B_z & -B_y \\ -E_y & -B_z & 0 & B_x \\ -E_z & B_y & -B_x & 0 \end{pmatrix}_{\mu\nu} \tag{2.9}$$

The equations of electrodynamics can often be put in more concise and symmetric form by also defining a dual field tensor $G^{\alpha\beta}$ as

$$G^{\alpha\beta} = \tilde{F}^{\alpha\beta} = \frac{1}{2}\varepsilon^{\alpha\beta\mu\nu}F_{\mu\nu} \tag{2.10}$$

The inverse relation can be found from eq.(2.6)

$$F^{\alpha\beta} = -\tilde{\tilde{F}}^{\alpha\beta} = -\tilde{G}^{\alpha\beta} = -\frac{1}{2}\varepsilon^{\alpha\beta\mu\nu}G_{\mu\nu} \tag{2.11}$$

In terms of the electric field $\mathbf{E}$ and the magnetic field $\mathbf{B}$, the dual field tensor is

$$G^{\alpha\beta} = \begin{pmatrix} 0 & B_x & B_y & B_z \\ -B_x & 0 & -E_z & E_y \\ -B_y & E_z & 0 & -E_x \\ -B_z & -E_y & E_x & 0 \end{pmatrix}_{\alpha\beta} \tag{2.12}$$

The standard Maxwell equations without magnetic charge then have the manifestly covariant form[2]

$$\partial_\alpha F^{\alpha\beta} = -\frac{1}{c}J^\beta \quad \text{and} \quad \partial_\alpha G^{\alpha\beta} = 0 \tag{2.13}$$

where J^α is the electric charge density four-vector with electric charge density ρ and flux density $\mathbf{J}$. The three-vector form of the standard Maxwell equations is

$$\boldsymbol{\nabla} \cdot \mathbf{E} = \rho \qquad \boldsymbol{\nabla} \times \mathbf{B} = +\frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} + \frac{1}{c}\mathbf{J} \tag{2.14}$$

$$\boldsymbol{\nabla} \cdot \mathbf{B} = 0 \qquad -\boldsymbol{\nabla} \times \mathbf{E} = \frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} \tag{2.15}$$

Inserting eq.(2.10) into the second of eq.(2.13) gives

$$\varepsilon^{\beta\alpha\mu\nu}\partial_\alpha F_{\mu\nu} = 0 \tag{2.16}$$

for all β values. This equation is equivalent to

$$\partial_\alpha F_{\mu\nu} + \partial_\mu F_{\nu\alpha} + \partial_\nu F_{\alpha\mu} = 0 \tag{2.17}$$

which is often quoted[3] as the covariant form of the so-called homogeneous Maxwell equations, eq.(2.15). However, the second of eq.(2.13) itself seems the preferable form since it shows clearly the absence of a magnetic charge density parallel to the electric charge density J^β.

[2] See eq.(12.126) of [5].
[3] See eq.(11.143) of [7].

3 Extended Electrodynamics with Electric and Magnetic Charge

In this section we present a synopsis of an *Extended Electrodynamics* with both electric and magnetic charges, and prove an extended electrodynamic version of the Poynting theorem.

Assuming that magnetic charge, if any, must be added to the Maxwell equations in a way that preserves Lorentz covariance, a conservative and plausible generalization of standard electrodynamics is simply to add a magnetic charge density four-vector as source of the dual field tensor in eq.(2.13).

The covariant Maxwell equations are then:

$$\partial_\alpha F^{\alpha\beta} = -\frac{1}{c} \mathsf{J}^\beta \qquad \partial_\alpha G^{\alpha\beta} = -\frac{1}{c} \mathsf{L}^\beta \tag{3.1}$$

where

$$\mathsf{J} = c\rho\mathbf{e}_0 + \mathbf{J} \qquad \mathsf{L} = c\lambda\mathbf{e}_0 + \mathbf{L} \tag{3.2}$$

are, respectively, the electric charge density four-vector J with electric charge density ρ and flux density $\mathbf{J}$ and the magnetic charge four-vector L with magnetic charge density λ and flux density $\mathbf{L}$.

In three-vector form, the extended Maxwell equations in eq.(3.1) are[4]

$$\nabla \cdot \mathbf{E} = \rho \qquad \nabla \times \mathbf{B} = \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} + \frac{1}{c}\mathbf{J} \tag{3.3}$$

$$\nabla \cdot \mathbf{B} = \lambda \qquad -\nabla \times \mathbf{E} = \frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} + \frac{1}{c}\mathbf{L} \tag{3.4}$$

It follows from eq.(3.1) that, due to the anti-symmetry of $F^{\alpha\beta}$ and $G^{\alpha\beta}$,

$$\partial_\beta \mathsf{J}^\beta = -c\partial_\alpha\partial_\beta F^{\alpha\beta} = 0 \qquad \partial_\beta \mathsf{L}^\beta = -c\partial_\alpha\partial_\beta G^{\alpha\beta} = 0 \tag{3.5}$$

In three-vector form these are the conservation rules for electric and magnetic charge

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{J} = 0 \qquad \frac{\partial \lambda}{\partial t} + \nabla \cdot \mathbf{L} = 0 \tag{3.6}$$

Another plausible generalization defines the extended Lorentz force density four-vector f^α by adding a comparable magnetic term f^α_{mg} to the standard electric term f^α_{el} so that

$$f^\alpha = \left(f^\alpha_{\mathrm{el}} + f^\alpha_{\mathrm{mg}}\right) \qquad \text{where} \qquad f^\alpha_{\mathrm{el}} = \frac{1}{c} F^\alpha{}_\gamma \mathsf{J}^\gamma \qquad f^\alpha_{\mathrm{mg}} = \frac{1}{c} G^\alpha{}_\gamma \mathsf{L}^\gamma \tag{3.7}$$

[4]Eqs.(3.3, 3.4) agree with eq.(6.150) of [7].

In three-vector form,

$$f^0_{\text{el}} = \frac{1}{c}\,(\mathbf{E}\cdot\mathbf{J}) \qquad\qquad f^0_{\text{mg}} = \frac{1}{c}\,(\mathbf{B}\cdot\mathbf{L}) \tag{3.8}$$

$$\mathbf{f}_{\text{el}} = \left\{\rho\mathbf{E} + \frac{1}{c}\,(\mathbf{J}\times\mathbf{B})\right\} \qquad\qquad \mathbf{f}_{\text{mg}} = \left\{\lambda\mathbf{B} - \frac{1}{c}\,(\mathbf{L}\times\mathbf{E})\right\} \tag{3.9}$$

We can also use eq.(3.1) to write the force density entirely in terms of field tensors

$$f^\alpha_{\text{el}} = \frac{1}{c}F^\alpha{}_\gamma J^\gamma = -F^\alpha{}_\gamma\left(\partial_\mu F^{\mu\gamma}\right) \tag{3.10}$$

$$f^\alpha_{\text{mg}} = \frac{1}{c}G^\alpha{}_\gamma L^\gamma = -G^\alpha{}_\gamma\left(\partial_\mu G^{\mu\gamma}\right) \tag{3.11}$$

Standard electrodynamics defines[5] the symmetric energy momentum tensor as

$$T^{\alpha\beta} = F^{\alpha\mu}F^{\beta\nu}\,\eta_{\mu\nu} - \frac{1}{4}\,\eta^{\alpha\beta}F_{\mu\nu}F^{\mu\nu} \tag{3.12}$$

This same definition also proves correct for extended electrodynamics.

Expansion of eq.(3.12) using matrix multiplication yields

$$T^{\alpha\beta} = T^{\beta\alpha} = \begin{pmatrix} \mathcal{E} & c\,P_x & c\,P_y & c\,P_z \\ c\,P_x & M_{11} & M_{12} & M_{13} \\ c\,P_y & M_{21} & M_{22} & M_{23} \\ c\,P_z & M_{31} & M_{32} & M_{33} \end{pmatrix}_{\alpha\beta} \tag{3.13}$$

where,

$$\mathcal{E} = \frac{1}{2}\left(E^2 + B^2\right) \qquad c\mathbf{P} = \mathbf{E}\times\mathbf{B} \qquad M_{ij} = -\left(E_iE_j + B_iB_j\right) + \delta_{ij}\mathcal{E} \tag{3.14}$$

The second term on the right in eq.(3.12) has the effect of making $T^{\alpha\beta}$ traceless. With the invariant trace defined as

$$\text{Tr}\,T = T^{\alpha\beta}\eta_{\alpha\beta} \tag{3.15}$$

it follows from eq.(3.12) that

$$\text{Tr}\,T = \eta_{\alpha\beta}\eta_{\mu\nu}F^{\alpha\mu}F^{\beta\nu} - F_{\mu\nu}F^{\mu\nu} = 0 \tag{3.16}$$

Using the identity eq.(2.7) with the substitutions $X^{\alpha\beta} = Y^{\alpha\beta} = F^{\alpha\beta}$, the standard definition in eq.(3.12) can also be written in an equivalent form

$$T^{\alpha\beta} = G^{\alpha\mu}G^{\beta\nu}\,\eta_{\mu\nu} - \frac{1}{4}\,\eta^{\alpha\beta}G_{\mu\nu}G^{\mu\nu} \tag{3.17}$$

[5] For example, see eq.(12.113) of [7].

Since the same substitutions show that $G_{\mu\nu}G^{\mu\nu} = -F_{\mu\nu}F^{\mu\nu}$, eqs.(3.12 and 3.17) can be added to give a third equivalent form

$$T^{\alpha\beta} = \frac{1}{2}\left\{F^{\alpha\mu}F^{\beta\nu}\,\eta_{\mu\nu} + G^{\alpha\mu}G^{\beta\nu}\,\eta_{\mu\nu}\right\} \tag{3.18}$$

Of these three equivalent forms, the third, eq.(3.18), is the simplest and most useful. It should be quoted in the textbooks rather than eq.(3.12).

As will be seen in Section 7, eq.(3.18) makes evident the invariance of $T^{\alpha\beta}$ under the dyality transformation. The otherwise accidental requirement that $T^{\alpha\beta}$ must be made traceless before it will reproduce the Poynting theorem can be understood as the requirement that electrodynamics must be invariant under the dyality transformation, and hence must be extended electrodynamics.

Using the equivalent definition of $T^{\alpha\beta}$ from eq.(3.18), together with eqs.(3.10, 3.11), Appendix A in Section 11 proves that

$$\partial_\alpha T^{\alpha\beta} = -\left(f^{\beta}_{\rm el} + f^{\beta}_{\rm mg}\right) = -f^{\beta} \tag{3.19}$$

which demonstrates both the correctness of the choice of $T^{\alpha\beta}$ in the equivalent eqs.(3.12, 3.17, and 3.18), and also the correctness of the force hypothesis in eq.(3.7).

For $\beta = 0$, eq.(3.19) expands to the Poynting theorem

$$\left(\frac{\partial \mathcal{E}}{\partial t} + \boldsymbol{\nabla}\cdot\mathbf{S}\right) = -\left(\mathbf{E}\cdot\mathbf{J}\right) - \left(\mathbf{B}\cdot\mathbf{L}\right) \tag{3.20}$$

where $\mathbf{S} = c\mathbf{E}\times\mathbf{B}$.

For $\beta = i$, where $i = 1, 2, 3$, eq.(3.19) expands to

$$\frac{\partial P_i}{\partial t} + \left(\boldsymbol{\nabla}\cdot\mathbb{M}\right)_i = -\left\{\rho\mathbf{E} + \frac{1}{c}\left(\mathbf{J}\times\mathbf{B}\right)\right\}_i - \left\{\lambda\mathbf{B} - \frac{1}{c}\left(\mathbf{L}\times\mathbf{E}\right)\right\}_i \tag{3.21}$$

where $\mathbf{P}$ and $\mathbb{M}$ are defined in eq.(3.14). Note that the sign of the three-dimensional dyadic $\mathbb{M}$ is defined here so that, with $d\mathbf{a}$ the outward pointing elements of surface $\mathcal{S}$, the integral $\oint_{\mathcal{S}}\left(\sum_{j=1}^{3} M_{ij}da_j\right)$ is the net *outgoing* flow of the $i^{\rm th}$ component of momentum.

4 Extended Maxwell Equations Derived from Two Vector Potentials

The extended Maxwell equations can be derived from two four-vector potentials, **A** and **N**. With the definition[6]

$$F^{\alpha\beta} = \left(\frac{\partial A^\beta}{\partial x^\alpha} - \frac{\partial A^\alpha}{\partial x^\beta}\right) - \varepsilon^{\alpha\beta\mu\nu}\left(\frac{\partial N^\nu}{\partial x^\mu} - \frac{\partial N^\mu}{\partial x^\nu}\right) \tag{4.1}$$

[6]Eqs.(4.1, 4.3, 5.5, 5.7) were first obtained by Shanmugadhasan[14]. See also [2]. For sign convention, compare eq.(11.136) of [7].

it follows that the dual field tensor is

$$G^{\alpha\beta} = \frac{1}{2}\varepsilon^{\alpha\beta\mu\nu}F_{\mu\nu} = \frac{1}{2}\varepsilon^{\alpha\beta\mu\nu}\left(\partial_\mu A_\nu - \partial_\nu A_\mu\right) - \frac{1}{2}\varepsilon^{\alpha\beta\mu\nu}\varepsilon_{\mu\nu\gamma\delta}\partial^\gamma N^\delta \tag{4.2}$$

Using the identity eq.(2.3), this is

$$G^{\alpha\beta} = \left(\frac{\partial N^\beta}{\partial x^\alpha} - \frac{\partial N^\alpha}{\partial x^\beta}\right) + \varepsilon^{\alpha\beta\mu\nu}\left(\frac{\partial A^\nu}{\partial x^\mu} - \frac{\partial A^\mu}{\partial x^\nu}\right) \tag{4.3}$$

With the definitions of the antisymmetric and gauge invariant tensors $a_{\alpha\beta}$ and $n_{\alpha\beta}$,

$$\begin{aligned} a_{\alpha\beta} &= \frac{\partial A_\alpha}{\partial x^\beta} - \frac{\partial A_\beta}{\partial x^\alpha} \\ n_{\alpha\beta} &= \frac{\partial N_\alpha}{\partial x^\beta} - \frac{\partial N_\beta}{\partial x^\alpha} \end{aligned} \tag{4.4}$$

and the definition of duals in eq.(2.5), the eqs.(4.1 and 4.3) may also be written as

$$\begin{aligned} F_{\alpha\beta} &= -a_{\alpha\beta} + \tilde{n}_{\alpha\beta} \\ G_{\alpha\beta} &= -n_{\alpha\beta} - \tilde{a}_{\alpha\beta} \end{aligned} \tag{4.5}$$

The potential four-vectors may be written with the notations $A^0 = \phi$ and $N^0 = \theta$

$$\mathsf{A} = \phi\mathbf{e}_0 + \mathbf{A} \qquad \mathsf{N} = \theta\mathbf{e}_0 + \mathbf{N} \tag{4.6}$$

Then the electric and magnetic fields can be written in terms of these potentials.

$$-E_i = F^{i0} = \partial^i A^0 - \partial^0 A^i - \varepsilon^{i0kl}\partial_k N_l \tag{4.7}$$

and thus

$$\mathbf{E} = -\boldsymbol{\nabla}\phi - \frac{1}{c}\frac{\partial\mathbf{A}}{\partial t} - \boldsymbol{\nabla}\times\mathbf{N} \tag{4.8}$$

Also

$$-B_i = G^{i0} = \partial^i N^0 - \partial^0 N^i + \varepsilon^{i0kl}\partial_k A_l \tag{4.9}$$

and thus

$$\mathbf{B} = -\boldsymbol{\nabla}\theta - \frac{1}{c}\frac{\partial\mathbf{N}}{\partial t} + \boldsymbol{\nabla}\times\mathbf{A} \tag{4.10}$$

5 Gauge Transformation of Potentials

A gauge transformation replaces the four-vector potentials A^α and N^α introduced in Section 4 by the modified potentials

$$A^{*\alpha} = A^\alpha + \partial^\alpha \Lambda \quad \text{and} \quad N^{*\alpha} = N^\alpha + \partial^\alpha \Gamma \tag{5.1}$$

where Λ and Γ are field functions. Then

$$F^{*\alpha\beta} = \left(\partial^\alpha A^{*\beta} - \partial^\beta A^{*\alpha}\right) - \varepsilon^{\alpha\beta\mu\nu}\partial_\mu N^*_\nu = F^{\alpha\beta} + \left(\partial^\alpha\partial^\beta - \partial^\beta\partial^\alpha\right)\Lambda - \frac{1}{2}\varepsilon^{\alpha\beta\mu\nu}\left(\partial_\mu\partial_\nu - \partial_\nu\partial_\mu\right)\Gamma = F^{\alpha\beta} \tag{5.2}$$

and similarly $G^{*\alpha\beta} = G^{\alpha\beta}$. The field tensors $F^{\alpha\beta}$and $G^{\alpha\beta}$, and thus the electric and magnetic fields **E** and **B**, are unchanged by gauge transformation of the potentials.

We assume the Lorenz conditions $\partial_\alpha A^\alpha = 0$ and $\partial_\alpha N^\alpha = 0$ for the four-vector potentials. The gauge-transformed potentials $A^{*\alpha}$ and $N^{*\alpha}$ are also four-vectors satisfying the Lorenz conditions $\partial_\alpha A^{*\alpha} = 0$ and $\partial_\alpha N^{*\alpha} = 0$ if and only if the field functions Λ and Γ are Lorentz scalar fields satisfying

$$\Box^2\Lambda = 0 \quad \text{and} \quad \Box^2\Gamma = 0 \tag{5.3}$$

which we also assume here.

Then the first of eq.(3.1) can be written as

$$-\frac{1}{c}J^\beta = \partial_\alpha F^{\alpha\beta} = \partial_\alpha\partial^\alpha A^\beta - \partial^\beta\left(\partial_\alpha A^\alpha\right) - \varepsilon^{\alpha\beta\gamma\delta}\partial_\alpha\partial_\gamma N_\delta \tag{5.4}$$

The last term vanishes due to antisymmetry and hence, assuming the Lorenz gauge condition, $\partial_\alpha A^\alpha = 0$,

$$\Box^2 A^\beta = -\frac{1}{c}J^\beta \tag{5.5}$$

Similarly, assuming the Lorenz gauge condition $\partial_\alpha N^\alpha = 0$ and the second of eq.(3.1),

$$-\frac{1}{c}L^\beta = \partial_\alpha G^{\alpha\beta} = \partial_\alpha\partial^\alpha N^\beta - \partial^\beta\left(\partial_\alpha N^\alpha\right) - \varepsilon^{\alpha\beta\gamma\delta}\partial_\alpha\partial_\gamma A_\delta \tag{5.6}$$

$$\Box^2 N^\beta = -\frac{1}{c}L^\beta \tag{5.7}$$

6 Formal Lagrangian Derivation of Extended Electrodynamics

This section shows that the Maxwell equations of extended electrodynamics can be derived from a Lagrangian field function.

Some approaches to a Lagrangian derivation of extended electrodynamics have been merely the starting point of an attempt to derive the Dirac monopole by merging Lagrangian electrodynamics with Dirac spinor sources.[7]

These earlier attempts at a Lagrangian for extended electrodynamics are constricted by their attempt to write the Lagrangian somehow as a sum of separate "electric" and "magnetic" Lagrangians $\mathcal{L} = \mathcal{L}_{\mathrm{el}} + \mathcal{L}_{\mathrm{mg}}$. Others[8] introduce "electric" fields $F^{\alpha\beta}$ that act only on electric charges and "magnetic" fields $F'^{\alpha\beta}$ that act only on magnetic charges.

The simplification here is, first, that the Lagrangian field $\mathcal{L}$ need not be separated into a symmetric sum of electric and magnetic Lagrangians.

The second simplification is that only one kind of electromagnetic field $F^{\alpha\beta}$ is needed. It can be *expressed* also in the form of a dual tensor $G^{\alpha\beta} = \tilde{F}^{\alpha\beta}$, but the **E** and **B** fields used in the two tensors are the same, just re-arranged. One must avoid conflating dyality and duality.

Although a formal Lagrangian derivation of the extended Maxwell equations proves possible, we must remember that application of Lagrangian methods to electromagnetism has a strong element of enlightened guesswork. We are loosely guided by analogy, and the precise choices $d/dt \to \partial/\partial x^\mu$, $u_k \to A_\alpha$, $v_k \to \Phi_{\alpha\mu} = \partial A_\alpha/\partial x^\mu$ suggested by mechanics must be justified by their success in practice.

The simplified Lagrangian chosen here is[9]

$$\mathcal{L} = -\frac{1}{4}F^{\alpha\beta}F_{\alpha\beta} - \frac{1}{c}J^\alpha A_\alpha + \frac{1}{c}L^\alpha N_\alpha \tag{6.1}$$

The role of "coordinates" is played by the potentials A_α and N_α, and the role of "generalized velocities" by $\Phi_{\alpha\beta}$ and $\Theta_{\alpha\beta}$ where

$$\Phi_{\alpha\beta} = \frac{\partial A_\alpha}{\partial x^\beta} \quad \text{and} \quad \Theta_{\alpha\beta} = \frac{\partial N_\alpha}{\partial x^\beta} \tag{6.2}$$

To express $\mathcal{L}$ in terms of these "coordinates" and "generalized velocities", write eq.(4.1) and eq.(4.3) as

$$F_{\alpha\beta} = -\left(\Phi_{\alpha\beta} - \Phi_{\beta\alpha}\right) + \varepsilon_{\alpha\beta}{}^{\mu\nu}\Theta_{\mu\nu} \qquad G_{\alpha\beta} = -\left(\Theta_{\alpha\beta} - \Theta_{\beta\alpha}\right) - \varepsilon_{\alpha\beta}{}^{\mu\nu}\Phi_{\mu\nu} \tag{6.3}$$

[7] For example, [14, 3, 2].

[8] See [13].

[9] See [11]. Since eq.(2.8) with $X{=}Y{=}F$ implies that $G^{\alpha\beta}G_{\alpha\beta} = -F^{\alpha\beta}F_{\alpha\beta}$, this Lagrangian can also be written in the more symmetric form $\mathcal{L} = -\left\{\frac{1}{8}F^{\alpha\beta}F_{\alpha\beta} + \frac{1}{c}J^\alpha A_\alpha\right\} + \left\{\frac{1}{8}G^{\alpha\beta}G_{\alpha\beta} + \frac{1}{c}L^\alpha N_\alpha\right\}$.

In taking partial derivatives, the Lagrangian field is considered as a function of the form $\mathcal{L} = \mathcal{L}(A, N, \Phi, \Theta, x)$ where $A \equiv [A_\alpha]$, $N \equiv [N_\alpha]$, $\Phi \equiv \left[\partial A_\alpha / \partial x_\mu\right]$, $\Theta \equiv \left[\partial N_\alpha / \partial x_\mu\right]$, and $x \equiv [x^\mu]$, and where the [ ] brackets denote the entire set of enclosed components. It follows that

$$\frac{\partial}{\partial \Phi_{\alpha\beta}} \left\{F^{\lambda\delta} F_{\lambda\delta}\right\} = -2 \left\{\frac{\partial}{\partial \Phi_{\alpha\beta}} \left(\Phi_{\lambda\delta} - \Phi_{\delta\lambda}\right)\right\} F^{\lambda\delta} = -4F^{\alpha\beta} \tag{6.4}$$

and

$$\frac{\partial}{\partial \Theta_{\alpha\beta}} \left\{F^{\lambda\delta} F_{\lambda\delta}\right\} = 2 \left\{\frac{\partial}{\partial \Theta_{\alpha\beta}} \left(\varepsilon_{\lambda\delta}{}^{\mu\nu} \Theta_{\mu\nu}\right)\right\} F^{\lambda\delta} = 2\varepsilon^{\lambda\delta\alpha\beta} F_{\lambda\delta} = 4G^{\alpha\beta} \tag{6.5}$$

Hence

$$\frac{\partial \mathcal{L}(A, N, \Phi, \Theta, x)}{\partial \Phi_{\alpha\beta}} = F^{\alpha\beta} \qquad \frac{\partial \mathcal{L}(A, N, \Phi, \Theta, x)}{\partial \Theta_{\alpha\beta}} = -G^{\alpha\beta} \tag{6.6}$$

The Lagrange equations, again chosen by analogy with mechanics, are

$$\partial_\beta \left(\frac{\partial \mathcal{L}}{\partial \Phi_{\alpha\beta}}\right) - \frac{\partial \mathcal{L}}{\partial A_\alpha} = 0 \qquad \partial_\beta \left(\frac{\partial \mathcal{L}}{\partial \Theta_{\alpha\beta}}\right) - \frac{\partial \mathcal{L}}{\partial N_\alpha} = 0 \tag{6.7}$$

Using eqs.(6.1 and 6.6), the Lagrange equations in eq.(6.7) expand to

$$\partial_\beta F^{\alpha\beta} - \frac{1}{c} J^\alpha = 0 \qquad - \partial_\beta G^{\alpha\beta} + \frac{1}{c} L^\alpha = 0 \tag{6.8}$$

Taking account of the anti-symmetry of $F^{\alpha\beta}$ and $G^{\alpha\beta}$, eq.(6.8) reproduces the covariant extended Maxwell equations in eq.(3.1).[10]

Note however that the Lagrangian analogy is limited. For example, the analogs of the generalized momenta defined in mechanics as $p_k = \partial L / \partial \dot{q}_k$, are the fields $F^{\alpha\beta}$ and $-G^{\alpha\beta}$ in eq.(6.6). But these are not independent; they are just the Maxwell field tensor and its dual. There is no covariant, extended electrodynamic analog of the Hamilton equations.

Attempts to derive the energy-momentum tensor $T^{\alpha\beta}$ from the Lagrangian are also unpersuasive. In Section 4.9 of [12], a symmetrizing correction term is added to a standard Lagrangian derivation of the energy-momentum tensor beginning from Noether's theorem. This method does give the correct $T^{\alpha\beta}$ for *standard* electrodynamics. The same method is quoted in [7] and [4]. However, the generalization of this method does not produce the correct $T^{\alpha\beta}$ for *extended* electrodynamics, even after a symmetrizing correction is added.

[10] The Lagrangian function $\mathcal{L}$ in eq.(6.1) is neither invariant under gauge transformation nor invariant under dyalitic transformations. This is admissible because, in both cases, the resulting Lagrange equations eq.(6.8) are invariant.

Also, a Lagrangian derivation of $T^{\alpha\beta}$ in §94 of [10], using Hamilton's principle together with variation of a general-relativistic metric, gives the correct value, but is formalistic and unpersuasive.

The most reliable verification of the energy-momentum tensor $T^{\alpha\beta}$ quoted in eq.(3.18) is the direct, algebraic proof of the Poynting theorem in Section 3 and Appendix A in Section 11.

7 The Dyality Transformation

This section defines the *Dyality Transformation* and shows *Dyality Invariance* under this transformation to be a symmetry only of extended electrodynamics.

For any solution to the source-free, standard Maxwell equations (eqs.(2.14, 2.15) with $\rho = 0$ and $\mathbf{J} = 0$) there is an alternate solution with primed fields defined as

$$\mathbf{E}' = \mathbf{B} \qquad \mathbf{B}' = -\mathbf{E} \tag{7.1}$$

With these definitions, eqs.(2.14, 2.15) give

$$-\boldsymbol{\nabla} \cdot \mathbf{B}' = 0 \qquad \boldsymbol{\nabla} \times \mathbf{E}' = -\frac{1}{c}\frac{\partial \mathbf{B}'}{\partial t} \tag{7.2}$$

$$\boldsymbol{\nabla} \cdot \mathbf{E}' = 0 \qquad \boldsymbol{\nabla} \times \mathbf{B}' = \frac{1}{c}\frac{\partial \mathbf{E}'}{\partial t} \tag{7.3}$$

which shows that the primed fields satisfy exactly the same Maxwell equations as eqs.(2.14, 2.15), but with primes on all fields.

However, that symmetry is broken when $\rho \neq 0$ or $\mathbf{J} \neq 0$. Then, for example, $-\boldsymbol{\nabla} \cdot \mathbf{B}' = \rho$ which is not a correct Maxwell equation. The transformation in eq.(7.1) is not a valid symmetry of standard electrodynamics when sources are present.

Symmetry can be regained by moving to the extended electrodynamics of Section 3, and including both fields and sources in the dyality transformation. Inclusion of A and N also guarantees dyality invariance of expressions involving these potentials. The *Dyality Transformation* is then defined as

$$\begin{pmatrix} \mathbf{E}' \\ \mathbf{B}' \end{pmatrix} = \begin{pmatrix} \mathbf{B} \\ -\mathbf{E} \end{pmatrix} \qquad \begin{pmatrix} \mathsf{J}' \\ \mathsf{L}' \end{pmatrix} = \begin{pmatrix} \mathsf{L} \\ -\mathsf{J} \end{pmatrix} \qquad \begin{pmatrix} \mathsf{A}' \\ \mathsf{N}' \end{pmatrix} = \begin{pmatrix} \mathsf{N} \\ -\mathsf{A} \end{pmatrix} \tag{7.4}$$

When these definitions are substituted into eq.(3.3) and eq.(3.4), the primed quantities then satisfy the same three-vector Maxwell equations as in Section 3

$$-\boldsymbol{\nabla} \cdot \mathbf{B}' = -\lambda' \qquad \boldsymbol{\nabla} \times \mathbf{E}' = -\frac{1}{c}\frac{\partial \mathbf{B}'}{\partial t} - \frac{1}{c}\mathbf{L}' \tag{7.5}$$

$$\nabla \cdot \mathbf{E}' = \rho' \qquad \nabla \times \mathbf{B}' = \frac{1}{c}\frac{\partial \mathbf{E}'}{\partial t} + \frac{1}{c}\mathbf{J}' \tag{7.6}$$

This is the dyality transformed solution of the Maxwell equations in which electric and magnetic quantities are exchanged.

This *Dyality Invariance* is a symmetry only of *Extended Electrodynamics,* with both electric and magnetic charges. As shown above, there is no such symmetry for standard electrodynamics except in the special source-free case.

Denote by $F'^{\alpha\beta}$ and $G'^{\alpha\beta}$ the same matrices as in eq.(2.9) and eq.(2.12), but written with primes on the electric and magnetic field components. Thus

$$F'^{\alpha\beta} = \begin{pmatrix} 0 & E'_x & E'_y & E'_z \\ -E'_x & 0 & B'_z & -B'_y \\ -E'_y & -B'_z & 0 & B'_x \\ -E'_z & B'_y & -B'_x & 0 \end{pmatrix}_{\alpha\beta} \tag{7.7}$$

with a similar expression for $G'^{\alpha\beta}$

$$G'^{\alpha\beta} = \begin{pmatrix} 0 & B'_x & B'_y & B'_z \\ -B'_x & 0 & -E'_z & E'_y \\ -B'_y & E'_z & 0 & -E'_x \\ -B'_z & -E'_y & E'_x & 0 \end{pmatrix}_{\alpha\beta} \tag{7.8}$$

Then substitution of the definitions in eq.(7.4) into eq.(7.7) and eq.(7.8) gives the dyality transformation of $F^{\alpha\beta}$ and $G^{\alpha\beta}$ as

$$\begin{pmatrix} F'^{\alpha\beta} \\ G'^{\alpha\beta} \end{pmatrix} = \begin{pmatrix} G^{\alpha\beta} \\ -F^{\alpha\beta} \end{pmatrix} \tag{7.9}$$

Substitute these into eq.(3.1) to obtain

$$-\partial_\mu G'^{\mu\nu} = \frac{1}{c} L'^\nu \quad \text{and} \quad \partial_\mu F'^{\mu\nu} = -\frac{1}{c} J'^\nu \tag{7.10}$$

in which the two equations in eq.(3.1) are merely interchanged. The primed quantities thus satisfy the same covariant Maxwell equations as those listed above in eq.(3.1). This is the covariant form of the alternate solution in eqs.(7.2 and 7.3).

We now consider the effect of the dyality transformation on the energy-momentum tensor $T^{\alpha\beta}$ as written in one of its equivalent forms in eq.(3.18). Let $T'^{\alpha\beta}$ denote the same matrix as the $T^{\alpha\beta}$ in eq.(3.18) but written with primes on the field components

$$T'^{\alpha\beta} = \frac{1}{2}\left\{ F'^{\alpha\mu} F'^{\beta\nu}\, \eta_{\mu\nu} + G'^{\alpha\mu} G'^{\beta\nu}\, \eta_{\mu\nu} \right\} \tag{7.11}$$

Substitution of eq.(7.9) into eq.(7.11) gives

$$T'^{\alpha\beta} = \frac{1}{2}\left\{F'^{\alpha\mu}F'^{\beta\nu}\,\eta_{\mu\nu} + G'^{\alpha\mu}G'^{\beta\nu}\,\eta_{\mu\nu}\right\} = \frac{1}{2}\left\{G^{\alpha\mu}G^{\beta\nu}\,\eta_{\mu\nu} + (-F^{\alpha\mu})\left(-F^{\beta\nu}\right)\eta_{\mu\nu}\right\} = T^{\alpha\beta} \tag{7.12}$$

The dyality transformation eq.(7.9) simply exchanges the two terms in eq.(3.18). Then $T^{\alpha\beta}$ is invariant under the dyality transformation, in the sense that the dyality substitution produces

$$T'^{\alpha\beta} = T^{\alpha\beta} \tag{7.13}$$

This invariance of $T^{\alpha\beta}$ can also be verified by inspection of eq.(3.13). Note that the dyality substitution makes

$$E'_iE'_j + B'_iB'_j = B_iB_j + (-E_i)\left(-E_j\right) = E_iE_j + B_iB_j \tag{7.14}$$

and hence $\mathcal{E}' = \mathcal{E}$ and $M'_{ij} = M_{ij}$. Also notice that $\mathbf{E}' \times \mathbf{B}' = \mathbf{B} \times (-\mathbf{E}) = \mathbf{E} \times \mathbf{B}$ and hence $P'_i = P_i$. Thus $T'^{\alpha\beta} = T^{\alpha\beta}$.

The dyality transformation must not be confused with the Lorentz transformation that sums over indices. The equation $T'^{\alpha\beta} = T^{\alpha\beta}$ holds independently for every matrix element of $T^{\alpha\beta}$. Thus eq.(7.13) implies that $T'^{00} = T^{00}$, $T'^{0i} = T^{0i}$, and so forth for the other indices.

8 The Dyality Structure of Extended Electrodynamics

One feature of the covariant form of the extended electrodynamics in Section 3 is the split of the equations into those governed by the field tensor $F^{\alpha\beta}$ with the electric charge density four-vector $\mathbf{J}$ on the one hand, and those governed by the dual field tensor $G^{\alpha\beta}$ with the magnetic charge density four-vector $\mathbf{L}$ on the other.

We see this division in the Maxwell equations in eq.(3.1) where

$$\partial_\alpha F^{\alpha\beta} = -\frac{1}{c}\,J^\beta \quad \text{and} \quad \partial_\alpha G^{\alpha\beta} = -\frac{1}{c}L^\beta \tag{8.1}$$

and in the Lorentz force densities in eq.(3.7) where $f^\alpha = (f^\alpha_{\mathrm{el}} + f^\alpha_{\mathrm{mg}})$ with

$$f^\alpha_{\mathrm{el}} = \frac{1}{c}F^\alpha{}_\gamma J^\gamma \quad \text{and} \quad f^\alpha_{\mathrm{mg}} = \frac{1}{c}G^\alpha{}_\gamma L^\gamma \tag{8.2}$$

This split also appears in the equations for the vector potentials $\mathbf{A}$ and $\mathbf{N}$ in eq.(5.5) and eq.(5.7) where

$$\Box^2 A^\beta = -\frac{1}{c}J^\beta \quad \text{and} \quad \Box^2 N^\beta = -\frac{1}{c}L^\beta \tag{8.3}$$

The four-vector potential **A** has only electric sources **J**, and the four-vector potential **N** has only magnetic sources **L**.

In each of the split cases, eqs.(8.1, 8.2, 8.3), the dyality transformation exchanges the two parts of the split, thus demonstrating the dyality invariance of the theory.

The energy momentum tensor $T^{\alpha\beta}$ in eq.(3.18) is a single expression written as a sum of quadratic terms in $F^{\alpha\beta}$ and $G^{\alpha\beta}$

$$T^{\alpha\beta} = \frac{1}{2}\left\{F^{\alpha\mu} F^{\beta\nu}\,\eta_{\mu\nu} + G^{\alpha\mu} G^{\beta\nu}\,\eta_{\mu\nu}\right\} \tag{8.4}$$

The dyality transformation exchanges these terms, thus producing the dyality invariance of $T^{\alpha\beta}$ seen in eq.(7.13).

The dyality transformation also exchanges the expressions in eq.(4.1) and eq.(4.3) for the Maxwell field $F^{\alpha\beta}$ and its dual $G^{\alpha\beta}$ in terms of the vector potentials; the eq.(4.1)

$$F^{\alpha\beta} = \left(\partial^\alpha A^\beta - \partial^\beta A^\alpha\right) - \varepsilon^{\alpha\beta\mu\nu}\partial_\mu N_\nu \tag{8.5}$$

becomes eq.(4.3)

$$G^{\alpha\beta} = \left(\partial^\alpha N^\beta - \partial^\beta N^\alpha\right) + \varepsilon^{\alpha\beta\mu\nu}\partial_\mu A_\nu \tag{8.6}$$

and vice-versa.

9 Does Dyality Invariance Imply a Magnetic Monopole?

Let us tentatively accept the hypothesis that a dyality invariant extended electrodynamics is the correct form of electrodynamics. We consider the implications of that hypothesis. In particular, what does it say about the reality or non-reality of magnetic charge?

The answer depends on the interpretation of the dyality transformation: active or passive.

What is called an *active* interpretation of the dyality transformation would imply the experimental existence of magnetic charge, and hence is currently ruled out.

The analogy here is with *active* rotations in standard three-dimensional vector algebra.[11] In active rotations, a three-vector $\mathbf{V}$ is transformed into a rotated vector $\mathbf{V}'$ whose components in the original coordinate system are given by $V_i' = \sum R_{ij}V_j$. The rotated vector $\mathbf{V}'$ has the same length but a different direction and hence is physically different.

[11] See Chapter 8 of [8]

Applying that analogy to eq.(7.4), in an active transformation the primed fields represent a different physical reality. Due to dyality invariance, they are still solutions to the Maxwell equations and the other equations of extended electrodynamics, but they are *alternate* solutions, not the same one.

For example, a plane, monochromatic, linearly polarized light wave in vacuum, with angular velocity ω and wave vector $\mathbf{k} = (\omega/c)\,\mathbf{e}_3$ has

$$\mathbf{E} = a\mathbf{e}_1 \cos\phi \qquad \mathbf{B} = a\mathbf{e}_2 \cos\phi \tag{9.1}$$

where $\phi = (kz - \omega t)$ and $z = x^3$. Apply eq.(7.4) to obtain

$$\mathbf{E}' = a\mathbf{e}_2 \cos\phi \qquad \mathbf{B}' = -a\mathbf{e}_1 \cos\phi \tag{9.2}$$

which is a plane wave linearly polarized in the $\mathbf{e}_2$ direction. The primed plane wave in eq.(9.2) is also a solution to Maxwell equations, but it is an *alternate* solution, physically distinguishable from the original solution in eq.(9.1).

For a more complex example, suppose that we begin with the extended Maxwell equations, but with $\mathbf{L} = 0$ since no magnetic charge has been found experimentally

$$\nabla \cdot \mathbf{E} = \rho \qquad \nabla \times \mathbf{B} = \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} + \frac{1}{c}\mathbf{J} \tag{9.3}$$

$$\nabla \cdot \mathbf{B} = 0 \qquad -\nabla \times \mathbf{E} = \frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} \tag{9.4}$$

Now apply the dyality transformation in eq.(7.4). The extended Maxwell equations then become

$$-\nabla \cdot \mathbf{B}' = -\lambda' \qquad \nabla \times \mathbf{E}' = -\frac{1}{c}\frac{\partial \mathbf{B}'}{\partial t} - \frac{1}{c}\mathbf{L}' \tag{9.5}$$

$$\nabla \cdot \mathbf{E}' = 0 \qquad \nabla \times \mathbf{B}' = \frac{1}{c}\frac{\partial \mathbf{E}'}{\partial t} \tag{9.6}$$

in which $\mathbf{L}' \neq 0$. Even if we were to begin with a solution with $\mathbf{L} = 0$, corresponding to the current experimental evidence that no magnetic charge exists, an active transformation such as eq.(7.4) would imply the real existence of an alternate solution in which $\mathbf{L}' \neq 0$. With the active interpretation of the dyality transformation, eqs.(9.5, 9.6) represent an experimentally real alternate solution in which magnetic charge is non-zero. Thus the active interpretation is ruled out currently by the failure to find those solutions experimentally.

However, a *passive* interpretation of dyalitic transformations can be used. In the *passive interpretation,* the primed quantities in eq.(7.4) are interpreted as representing the same physical reality as the unprimed ones, just viewed differently.

The analogy here is with passive rotations in standard three-dimensional vector algebra. The coordinate system is rotated in the opposite sense, and the vector components transform as before, by $V_i' = \sum R_{ij} V_j$. But the vector $\mathbf{V}$ itself is unchanged. Vector $\mathbf{V}$ has different components only because it is being viewed now from a different observer orientation.

For example, the passive interpretation holds eq.(9.1) and eq.(9.2) to represent the same physical reality. The only change is that quantities previously denoted $\mathbf{E}$, $\mathbf{B}$ are now being denoted by $-\mathbf{B}'$, $\mathbf{E}'$. In the other example, with the passive interpretation of this dyality transformation, eqs.(9.5, 9.6) represent the same physical reality as eqs.(9.3, 9.4). The only change is that quantities previously denoted $\mathbf{E}$, $\mathbf{B}$, J are now being denoted by $-\mathbf{B}'$, $\mathbf{E}'$, $-\mathsf{L}'$.

The passive interpretation is evidently used by Jackson[12] when he says that, if all[13] particles have the same ratio of magnetic to electric charge, then we are free to use the generalized definition of dyalitic transformation in Appendix B and choose χ in eq.(12.1) such that $\mathsf{L}' = 0$. Call that ratio $r = q_m/q_e$, and use the second of eq.(12.1) to write

$$\mathsf{L}' = \mathsf{J}\,(-\sin\chi + r\cos\chi) \tag{9.7}$$

and then choose $\tan\chi = r$ to make $\mathsf{L}' = 0$. Then we "... have the Maxwell equations as they are usually known." In that case, "...it is a matter of convention to speak of a particle possessing an electric charge, but not magnetic charge."

And, at present there is indeed a universally accepted value for the ratio r of magnetic to electric charge of all particles. It is $r = 0$. Thus, viewing the situation using Jackson's language, we are free to begin with $\mathsf{L} = 0$ and use $r = 0$ and $\chi = 0$ (the identity transformation) to maintain $\mathsf{L}' = 0$ uniformly and consistently. Just as we choose our coordinate system for convenience in standard vector calculus, we are free to choose $\chi = 0$ for our convenience. The result is still extended electrodynamics, but with its passive dyality symmetry used to justify maintaining L' equal to zero. Thus the standard form of Maxwell equations in eqs.(9.3, 9.4) can be considered as extended Maxwell equations with passive dyality symmetry used to maintain $\chi = 0$ and $\mathsf{L} = 0$ universally.

However, if a particle with a magnetic charge ratio $|q_m/q_e|$ greater than the near-zero value of the electron is eventually found, then there will no longer be a *universal* value of r. It will have a near-zero value for the electron, and a presumably larger value for the magnetically charged particle. Then passive dyality invariance can no longer be used to keep $\mathsf{L}' = 0$ universally.

Thus, if some future magnetic-charge-detection experiment can be described correctly by the extended Maxwell equations with zero magnetic charge for the electron but nonzero magnetic charge for some other

[12] Section 6.12, p.252 of [7].

[13] Note the word *all*; universality is required. Clearly, such a scheme will only be coherent if it is universal. All of physics would have to agree on the chosen χ value.

particle, then there will no longer be a passive dyality transformation that transforms away the magnetic charge sources. If nonzero magnetic charge exists, experimental evidence for its existence cannot be hidden by a passive dyality transformation.

10 Afterword

It follows from eq.(3.20) that the Poynting vector $\mathbf{S} = c\mathbf{E} \times \mathbf{B}$ is the flux density of the field energy $\mathcal{E} = \frac{1}{2}\left(E^2 + B^2\right)$. If one were to suppose that electromagnetic energy flows as a classical fluid, then there would be a velocity $\mathbf{V}(x)$ defined at each field point such that[14]

$$\mathbf{S} = \mathcal{E}\mathbf{V} \tag{10.1}$$

If that were the case, then dividing by c^2 and using the definition of field momentum $\mathbf{P}$ from eq.(3.14) would give

$$\mathbf{P} = \mathcal{M}\mathbf{V} \tag{10.2}$$

where $\mathcal{M} = \mathcal{E}/c^2$ is a relativistic mass density. We could then conclude that field momentum is due to the flow of field mass.

However, it has been proved[15] that there is no special relativistically correct velocity $\mathbf{V}$ satisfying eq.(10.1). This is because the conservation of electromagnetic energy-momentum is expressed as the divergence of a symmetric four-tensor $T^{\alpha\beta}$ rather than as the divergence of a (non-existent) four-vector.

Thus eq.(10.2) is not true; electromagnetic field momentum cannot be explained as due to the flow of relativistic mass. And yet electromagnetic field momentum is freely exchangeable with particle momentum. If electromagnetic field momentum is not due to the flow of field energy, then what is it? And what is the particle momentum with which it can be freely exchanged?

Attempts to answer those questions may provide clues to a future quantum mechanics.[16] If so, we should concentrate on a deeper understanding of electrodynamics, as this paper attempts to encourage.

Also, the deficiencies of the formal Lagrangian derivation of extended electrodynamics noted in Section 6 suggest that future fundamental electrodynamic research should concentrate on the extended Maxwell equations themselves and not on their Lagrangian derivation.

[14] See Chapter 2 of [1].

[15] See [9].

[16] For example, the non-existence of $\mathbf{V}(x)$ in eq.(10.2) may relate to the non-existence of velocity eigenstates in quantum mechanics.

11 Appendix A:The Poynting Theorem

To prove that $\partial_\alpha T^{\alpha\beta} = -f^\beta = -\left(f_{\mathrm{el}}^\beta + f_{\mathrm{mg}}^\beta\right)$.

From eq.(3.18), this is to prove that

$$-\left(f_{\mathrm{el}}^\beta + f_{\mathrm{mg}}^\beta\right) = \partial_\alpha T^{\alpha\beta} = \frac{1}{2}\partial_\alpha \left\{F^{\alpha\mu} F^{\beta\nu}\,\eta_{\mu\nu} + G^{\alpha\mu} G^{\beta\nu}\,\eta_{\mu\nu}\right\} \tag{11.1}$$

which, using eq.(3.10) and eq.(3.11) and the product rule, is equivalent to

$$\left\{F^\beta{}_\gamma\left(\partial_\alpha F^{\alpha\gamma}\right) + G^\beta{}_\gamma\left(\partial_\alpha G^{\alpha\gamma}\right)\right\} = \frac{1}{2}\left\{F^\beta{}_\gamma\left(\partial_\alpha F^{\alpha\gamma}\right) + F^\alpha{}_\gamma\left(\partial_\alpha F^{\beta\gamma}\right) + G^\beta{}_\gamma\left(\partial_\alpha G^{\alpha\gamma}\right) + G^\alpha{}_\gamma\left(\partial_\alpha G^{\beta\gamma}\right)\right\} \tag{11.2}$$

Thus, we must prove that

$$F^\beta{}_\gamma\left(\partial_\alpha F^{\alpha\gamma}\right) + G^\beta{}_\gamma\left(\partial_\alpha G^{\alpha\gamma}\right) = F^\alpha{}_\gamma\left(\partial_\alpha F^{\beta\gamma}\right) + G^\alpha{}_\gamma\left(\partial_\alpha G^{\beta\gamma}\right) \tag{11.3}$$

Defining

$$\phi_\beta \equiv G_{\beta\gamma}\left(\partial_\alpha G^{\alpha\gamma}\right) - G^{\alpha\gamma}\left(\partial_\alpha G_{\beta\gamma}\right) \tag{11.4}$$

we must prove that

$$\phi_\beta = -F_{\beta\gamma}\left(\partial_\alpha F^{\alpha\gamma}\right) + F^{\alpha\gamma}\left(\partial_\alpha F_{\beta\gamma}\right) \tag{11.5}$$

Inserting eq.(2.10) into eq.(11.4) gives

$$\phi_\beta = \frac{1}{4}\varepsilon_{\gamma\beta\theta\delta} F^{\theta\delta}\left(\partial_\alpha \varepsilon^{\gamma\alpha\mu\nu} F_{\mu\nu}\right) - \frac{1}{4}\varepsilon^{\gamma\alpha\mu\nu} F_{\mu\nu}\left(\partial_\alpha \varepsilon_{\gamma\beta\theta\delta} F^{\theta\delta}\right)$$

Using the identity eq.(2.4), this may be written

$$\phi_\beta = \frac{1}{4}\Big\{-\left(\delta^\alpha_\beta\delta^\mu_\theta\delta^\nu_\delta + \delta^\alpha_\theta\delta^\mu_\delta\delta^\nu_\beta + \delta^\alpha_\delta\delta^\mu_\beta\delta^\nu_\theta\right) + \left(\delta^\alpha_\beta\delta^\mu_\delta\delta^\nu_\theta + \delta^\alpha_\delta\delta^\mu_\theta\delta^\nu_\beta + \delta^\alpha_\theta\delta^\mu_\beta\delta^\nu_\delta\right)\Big\}\left\{F^{\theta\delta}\partial_\alpha F_{\mu\nu} - F_{\mu\nu}\partial_\alpha F^{\theta\delta}\right\} \tag{11.6}$$

$$\phi_\beta = -\frac{1}{2}\left(\delta^\alpha_\beta\delta^\mu_\theta\delta^\nu_\delta + \delta^\alpha_\theta\delta^\mu_\delta\delta^\nu_\beta + \delta^\alpha_\delta\delta^\mu_\beta\delta^\nu_\theta\right)\left(F^{\theta\delta}\partial_\alpha F_{\mu\nu} - F_{\mu\nu}\partial_\alpha F^{\theta\delta}\right) \tag{11.7}$$

$$\phi_\beta = -\frac{1}{2}\left(F^{\mu\nu}\partial_\beta F_{\mu\nu} + F^{\alpha\mu}\partial_\alpha F_{\mu\beta} + F^{\nu\alpha}\partial_\alpha F_{\beta\nu}\right) + \frac{1}{2}\left(F_{\mu\nu}\partial_\beta F^{\mu\nu} + F_{\mu\beta}\partial_\alpha F^{\alpha\mu} + F_{\beta\nu}\partial_\alpha F^{\nu\alpha}\right) \tag{11.8}$$

$$\phi_\beta = -F_{\beta\gamma}\left(\partial_\alpha F^{\alpha\gamma}\right) + F^{\alpha\gamma}\left(\partial_\alpha F_{\beta\gamma}\right) \tag{11.9}$$

which is eq.(11.5), as was to be proved.

12 Appendix B: Generalized Dyality Transformation

The definition of dyality transformation in eq.(7.4) can be generalized to

$$\begin{aligned}
\begin{pmatrix} \mathbf{E}' \\ \mathbf{B}' \end{pmatrix} &= \begin{pmatrix} \cos\chi & \sin\chi \\ -\sin\chi & \cos\chi \end{pmatrix} \begin{pmatrix} \mathbf{E} \\ \mathbf{B} \end{pmatrix} \\
\begin{pmatrix} \mathsf{J}' \\ \mathsf{L}' \end{pmatrix} &= \begin{pmatrix} \cos\chi & \sin\chi \\ -\sin\chi & \cos\chi \end{pmatrix} \begin{pmatrix} \mathsf{J} \\ \mathsf{L} \end{pmatrix} \\
\begin{pmatrix} \mathsf{A}' \\ \mathsf{N}' \end{pmatrix} &= \begin{pmatrix} \cos\chi & \sin\chi \\ -\sin\chi & \cos\chi \end{pmatrix} \begin{pmatrix} \mathsf{A} \\ \mathsf{N} \end{pmatrix}
\end{aligned} \tag{12.1}$$

where χ is a constant parameter.[17] The earlier definition in eq.(7.4) is the special case with $\chi = \pi/2$. Due to the linearity and orthogonality of eq.(12.1), all dyality invariance arguments given earlier in the paper using $\chi = \pi/2$ are also valid for general χ. For example, eq.(7.9) generalizes to

$$\begin{pmatrix} F'^{\alpha\beta} \\ G'^{\alpha\beta} \end{pmatrix} = \begin{pmatrix} \cos\chi & \sin\chi \\ -\sin\chi & \cos\chi \end{pmatrix} \begin{pmatrix} F^{\alpha\beta} \\ G^{\alpha\beta} \end{pmatrix} \tag{12.2}$$

Also $T'^{\alpha\beta} = T^{\alpha\beta}$ and the Poynting theorem in eq.(3.19) is preserved, regardless of the χ value.

[17] See eqs.(6.151 and 6.152) of [7].

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